Quantum Field Theory

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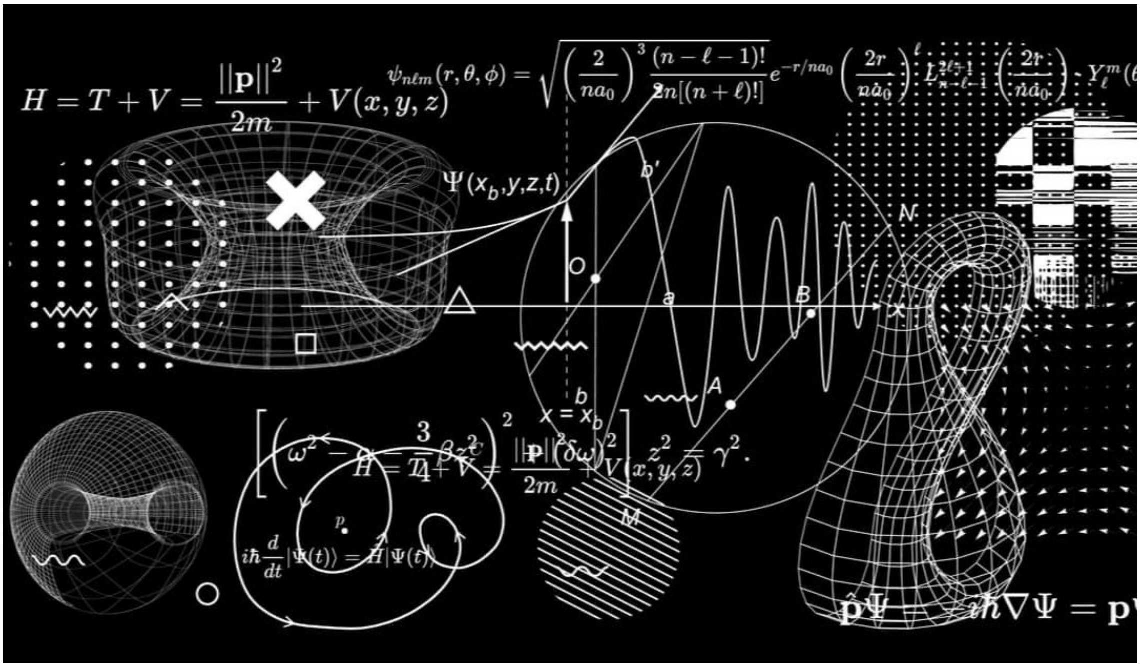
Abstract

The report contains an overview of what I have learnt in Quantum Field Theory during Summer of Science 2021, IIT Bombay.

Introduction

Quantum Field Theory emerged out of an attempt to club Quantum Mechanics and Einstein's Theory of Relativity. It now appears to be the most fundamental theory in physics, being a positive candidate for "The Theory of Everything", a consequence of which is String Theory, Quantum Gravity, Elementary Particle Physics, Quantum Chromodynamics, The Standard Model and many more.



1 Theory of Fields

1.1 Scalar Field

Functions of space and time that are Lorentz invariant. *eg.* Temperature of a fluid.

$$\phi(x) \rightarrow \phi(x)$$

1.2 Vector Field

Transform like 4-vectors under Lorentz transformations. It is common to denote 3-vectors by $\vec{x}$ and 4-vectors by x.

1.3 Tensor Field

Transform like tensors of some rank under Lorentz transformations. The quantity $v^\mu v_\mu$ is constant in all frames. *eg.* d'Alembertian $\Box$ is the simplest Lorentz invariant operator; parity and time-reversal operators are Lorentz invariant operators.

1.4 Free massless fields

They always satisfy the equation

$$\Box\phi = 0.$$

General solution is

$$\phi(x,t) = \int \frac{1}{(2\pi)^3}(a_p(t)e^{\iota\vec{p}\cdot\vec{x}} + a_p^+(t)e^{-\iota\vec{p}\cdot\vec{x}})\, d^3p$$

with

$$(\partial_t^2 + \vec{p}\cdot\vec{p})a_p(t) = 0.$$

One particular solution is

$$\phi(x) = a_p(t)e^{\iota\vec{p}\cdot\vec{x}}.$$

a_p^+ is called the *creation operator* and a_p is called the *annihilation operator* as

$$a_p^+|0\rangle = |\vec{p}\rangle.$$

1.5 Time dependance of creation and annihilation operators

Working in the Heisenberg picture, we have

$$a_p(t) = e^{-\iota\omega_p t}a_p$$
$$a_p^+(t) = e^{\iota\omega_p t}a_p^+.$$

2 The Klein-Gordon Equation

It is the following differential equation

$$(\Box + m^2)\phi = 0.$$

We get this by minimizing the action

$$S = \int [\frac{1}{2}\partial_\mu\phi\partial_\mu\phi - \nu(\phi)]\, d^4x$$

where $\nu(\phi)$ is the potential energy/*interaction energy density* term. The integrand is the simplest possible Lagrangian as it has no self-interacting terms.

3 Noether's Theorem

If a Lagrangian has a *continuous symmetry* then there exists a *current* associated with the symmetry that is conserved when the equations of motion are satisfied.

3.1 An Example

Lagrangian

$$L = |\partial_\mu \phi|^2 - m^2 |\phi|^2$$

L is invariant under $\phi \rightarrow e^{-\iota\alpha}\phi$ for any $\alpha \in \Re$. We can write

$$L = \partial_\mu \phi \partial_\mu \phi^* - m^2 \phi \phi^*.$$

We get two equations of motion (or rather field solutions) that are

$$(\Box + m^2)\phi = 0$$

$$(\Box + m^2)\phi^* = 0.$$

The equations of motion reduce to $\partial_\mu J_\mu = 0$ where

$$J_\mu = \sum_n \frac{\partial L}{\partial(\partial_\mu \phi_n)} \frac{\partial \phi_n}{\partial \alpha}$$

that is the *Noether current*. We can derive Coulomb's Potential Law using these simple ideas.

4 The S-matrix

A framework to describe interactions.

4.1 Time-Ordered Product

Operator with later time value must be put to left

$$\Im[A(x), B(x')] \equiv \begin{cases} A(x)B(x') & t > t' \\ B(x')A(x) & t' > t \end{cases}$$

4.2 Evolution Operator

We have

$$H = H_0 + H_I$$

Define evolution operator $U_0(t)$ as

$$|\Psi_0(t)\rangle = U_0(t)\Psi_0(-\infty) \equiv U_0(t)|i\rangle.$$

Define evolution operator $U(t)$ as

$$|\Psi(t)\rangle = U_0(t)U(t)U_0^+(t)\Psi_0(t).$$

Here Ψ_0 is the solution to the free Hamiltonian and ψ is the solution to actual Hamiltonian. Substituting this in time-dependant Schrödinger equation we get

$$U_0(t) = e^{-\iota H_0 t}$$

assuming H_0 is independent of time (as is seen in the Heisenberg picture). Similarly for the full Hamiltonian, putting boundary conditions such as $H_I \to 0$ as $t \to -\infty$ we get

$$U(t) = 1 + \sum_{n=1}^{\infty} \frac{(-\iota)^n}{n!} \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2 ..... \int_{-\infty}^{t} dt_n \Im[H_I(t_1) H_I(t_2) ..... H_I(t_n)]$$

compactly writing it as

$$U(t) = \Im[e^{-\iota \int_{-\infty}^{t} H_I(t')\, dt'}]$$

where $\Im$ is the *time ordering product*. In most cases this series converges. The S-matrix is unitary for any t, *i.e.*, $SS^+ = S^+ S = 1$. Transition amplitude for $|i\rangle \to |f\rangle$ at late times $t \to \infty$ is $\langle f|S|i\rangle$.

4.3 An Example - Meson Decay

$|i\rangle = \sqrt{2E_p} a_p^+ |0\rangle$ $|f\rangle = \sqrt{4E_{q_1} E_{q_2}} b_{q_1}^+ c_{q_2}^+ |0\rangle$ Initial state consists of a single meson of momentum p. Final state consists of nucleon-antinucleon pair of momentum q_1 and q_2. Amplitude frequency for decay of meson to nucleon-antinucleon pair is

$$\langle f|S|i\rangle = -\iota g \langle f| \int \Psi^+(x) \Psi(x) \Phi(x) |i\rangle.$$

We can use that $\Phi \simeq a + a^+$, $\Psi \simeq b + c^+$ and $\Psi^+ \simeq b^+ + c$. So we get

$$\langle f|S|i\rangle = -\iota g \langle 0| \int \int \frac{\sqrt{E_{q_1} E_{q_2}}}{\sqrt{E_{k_1} E_{k_2}}} a_{q_2} b_{q_1} a_{k_1}^+ b_{k_2}^+ e^{\iota (k_1 + k_2 - p)\cdot x} \frac{d^4x d^3k_1 d^3k_2}{(2\pi)^6}$$

$$= -\iota g (2\pi)^4 \delta^4(q_1 + q_2 - p).$$

5 Tools for Feynman diagrams

5.1 Normal Ordering

Normal ordered string of operators $\Phi_1(x_1)\Phi_2(x_2).....\Phi_n(x_n)$ is written as

$$: \Phi_1(x_1)\Phi_2(x_2).....\Phi_n(x_n) : .$$

It is defined to be the usual product with the annihilation operators on the right.

5.2 Propagators

> *"If we prepare a particle at spacetime point y, what is the amplitude of finding it at point x?"*

$$\langle 0|\Phi(x)\Phi(y)|0\rangle = \int \int \frac{d^3p d^3p'}{(2\pi)^6} \frac{1}{\sqrt{4E_p E_{p'}}} \langle 0|a_p a_p^+|0\rangle e^{-\iota p\cdot x + \iota p'\cdot y}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-\iota p\cdot(x-y)} \equiv D(x-y).$$

The function $D(x - y)$ is called propagator.

5.3 The Feynman Propagator

$$\Delta_F(x-y) \equiv \langle 0|\Im\Phi(x)\Phi(y)|0\rangle = \begin{cases} D(x-y) & x^0 > y^0 \\ D(y-x) & x^0 < y^0 \end{cases}$$

where $\Im$ stands for time-ordering. Δ_F can also be written as

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2}e^{-\imath p\cdot(x-y)}$$

where $p^2 = (p^0)^2 - \vec{p}^2$. This can be simplified as
for $x^0 > y^0$ we have

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-\imath p\cdot(x-y)},$$

and for $y^0 > x^0$ we have

$$D(y-x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-\imath p\cdot(y-x)}$$

where $E_p = p_0$.

5.4 Contraction

Contraction of $\Phi(x)\Phi(y)$ is defined as

$$\overbrace{\Phi(x)\Phi(y)} = \Delta_F(x-y)$$

i.e. replacing the operators with the Feynman propagator. Another definition of contraction, that is more intuitive, is

$$\langle 0|\Im[\Phi(x)\Phi(y)]|0\rangle \equiv \underbrace{\Phi(x)\Phi(y)}$$

We also define

$$\overbrace{\Psi(x)\Psi^+(y)} = \Delta_F(x-y)$$

$$\overbrace{\Psi(x)\Psi(y)} = \overbrace{\Psi(x)^+\Psi^+(y)} = 0.$$

5.5 Wick's Theorem

For any collection of fields $\Phi_1 \equiv \Phi(x_1)$, $\Phi_2 \equiv \Phi(x_2)$, etc we have

$$\Im(\Phi_1\Phi_2.....\Phi_n) =: \Phi_1\Phi_2.....\Phi_n : + : \text{all possible contractions} : .$$

For example, for $n = 2$

$$\Im\Phi(x)\Phi(y) = \ :\Phi(x)\Phi(y): + \Delta_F(x-y).$$

For example, for $n = 4$

$$\begin{aligned}\Im\Phi_1\Phi_2\Phi_3\Phi_4 = \ & :\Phi_1\Phi_2\Phi_3\Phi_4: + \overbrace{\Phi_1\Phi_2}:\Phi_3\Phi_4: + \overbrace{\Phi_1\Phi_3}:\Phi_2\Phi_4: \\ & + (\text{4 similar terms}) + \overbrace{\Phi_1\Phi_2}\overbrace{\Phi_3\Phi_4} + \overbrace{\Phi_1\Phi_3}\overbrace{\Phi_2\Phi_4} + \overbrace{\Phi_1\Phi_4}\overbrace{\Phi_2\Phi_3}.\end{aligned}$$

6 Feynman diagrams

They give the expansion for $\langle f|S-1|i\rangle$.

- Draw an external line for each particle in initial state $|i\rangle$ and each particle in final state $|f\rangle$. Choose dotted lines for mesons, solid lines for nucleons. Assign a directed momentum p to each line. Add an arrow to solid lines to denote its charge - incoming arrow in the initial state for $\bar{\Psi}$. For final state, outgoing arrow indicates Ψ and incoming arrow indicates $\bar{\Psi}$.

- Join the external lines together with trivalent vertices.

- Add a momentum k to each interval line.

- To each vertex write down a factor of $-\iota g(2\pi)^4\delta^4(\sum k)$ where $\sum k$ is the sum of all momenta flowing "into" the vertex.

- For each dotted line corresponding to a Φ particle with momentum k, write down a factor of
$$\int \frac{d^4k}{(2\pi)^4}\frac{\iota}{k^2-m^2+\iota\epsilon}$$
Include the same factor for solid internal lines with m replaced by nucleon mass M.

- Draw all possible diagrams with appropriate external legs and impose 4-momentum conservation at each vertex.

- Write down a factor of $-\iota g$ at each vertex.

- For each line write down the propagator.

- Integrate over momentum k flowing through each loop $\int \frac{d^4k}{(2\pi)^4}$.

$\langle f|S-1|i\rangle = \iota A_{fi}(2\pi)^4\delta^4(p_f-p_i)$ where A_{fi} is called the *scattering amplitude.* Now we will look at some real applications of Feynman diagrams (upto lowest order in S-matrix element) and how they simplify our task.

6.1 Nucleon Scattering

$$\Psi\Psi \to \Psi\Psi$$

$$A_{fi} = (-\iota g)^2[\frac{1}{(p_1-p_1')^2-m^2}+\frac{1}{(p_1-p_2')^2-m^2}]$$

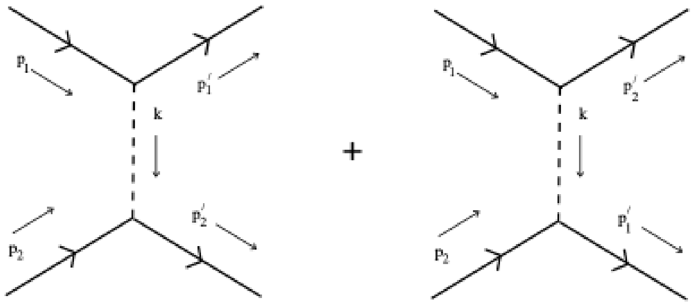
Here $k = p_1-p_1' = p_2'-p_2$ is the momentum of the *"virtual" meson* that is exchanged by the both nucleons. *("mes" in Greek means 'middle')*

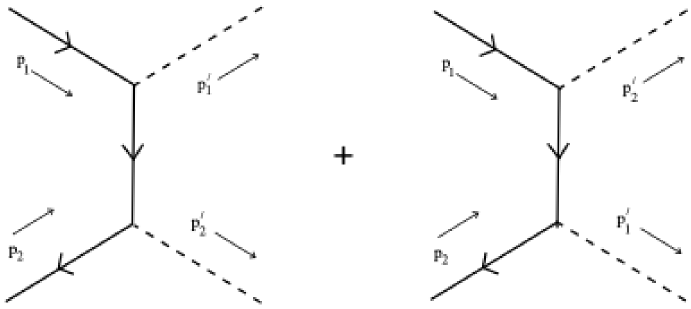
6.2 Nucleon to Meson Scattering

$$\Psi\bar{\Psi} \to \Phi\Phi$$

$$A_{fi} = (-\iota g)^2[\frac{1}{(p_1-p_1')^2-M^2}+\frac{1}{(p_1-p_2')^2-M^2}]$$

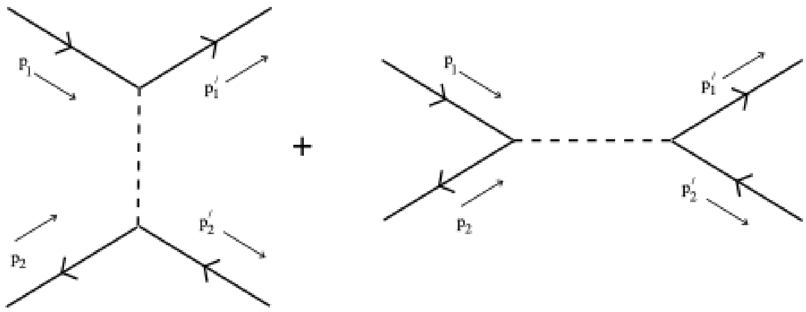
Here the virtual particle is a nucleon.





6.3 Nucleon-Antinucleon Scattering

$$\Psi\bar{\Psi} \to \Psi\bar{\Psi}$$

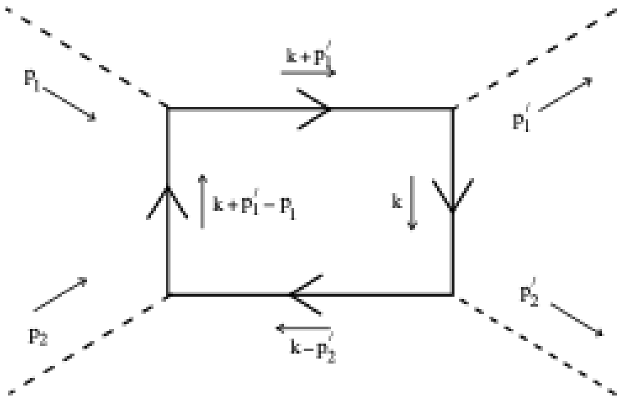


$$A_{fi} = (-\iota g)^2 [\frac{1}{(p_1 - p_1')^2 - m^2} + \frac{1}{(p_1 + p_2')^2 - m^2 + \iota\epsilon}]$$

In the COM frame, denominator of second term becomes $4(M^2 + p_1^2) - m^2 + 0$. So we get resonance at $4(M^2 + p_1^2) = m^2$ which is an opportunity to discover new particles.

6.4 Meson Scattering

$$\Phi\Phi \to \Phi\Phi$$



$$A_{fi} = (-\iota g)^4 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - M^2 + \iota\epsilon)((k+p_1')^2 - M^2 + \iota\epsilon)} \frac{1}{(k+p_1'-p_1)^2 - M^2 + \iota\epsilon)((k-p_2')^2 - M^2 + \iota\epsilon)}$$

For large k the integral goes as $\int \frac{d^4k}{k^8}$ which is convergent. But k need not always be large.

Solved problems

1 Classical formulation of three coupled fields

Consider the interaction term in the following Lagrangian density, for 3 scalar fields ϕ_1, ϕ_2, ϕ_3 associated to spinless particles of masses m_1, m_2, m_3,

$$L_I(x) = -\kappa\phi_1(x)\phi_2(x)\phi_3(x).$$

Note that magnitude of κ gives the strength of interaction.
(a) Show that this Lagrangian part is Hermitian for real κ.
(b) Find the Hamiltonian density H_I.

Sol. (a)

$$\begin{aligned} L_I^\dagger = (-\kappa\phi_1\phi_2\phi_3)^\dagger &= -\phi_3^\dagger\phi_2^\dagger\phi_1^\dagger\kappa^\dagger = -\phi_3\phi_2\phi_1\kappa \\ &= -\kappa\phi_1\phi_2\phi_3 \end{aligned}$$

(b)

$$\begin{aligned} H_I = \sum p_i\dot{q}_i - L &= \sum \frac{\partial L}{\partial\dot{\phi}_i}\dot{\phi}_i - L \\ &= \kappa\phi_1(x)\phi_2(x)\phi_3(x) = H_I(x) \end{aligned}$$

2 Wick's Theorem

The above Lagrangian results in the second order term of time series to have the product $\Im[:\phi_1\phi_2\phi_3|_{x_1}\phi_1\phi_2\phi_3|_{x_2}:]$. Apply Wick's Theorem to express this in terms of product of 3 fields.

Sol.

$$\Im[:\phi_1\phi_2\phi_3|_{x_1}\phi_1\phi_2\phi_3|_{x_2}:] =: \phi_1\phi_2\phi_3|_{x_1}\phi_1\phi_2\phi_3|_{x_2} : +$$

$$\overbrace{\phi_1(x_1)\phi_1(x_2)} : \phi_2(x_1)\phi_2(x_2)\phi_3(x_1)\phi_3(x_2) : + \overbrace{\phi_2(x_1)\phi_2(x_2)} : \phi_1(x_1)\phi_1(x_2)\phi_3(x_1)\phi_3(x_2) :$$

$$+ \overbrace{\phi_3(x_1)\phi_3(x_2)} : \phi_1(x_1)\phi_1(x_2)\phi_2(x_1)\phi_2(x_2) :$$

$$+ \overbrace{\phi_1(x_1)\phi_1(x_2)}\overbrace{\phi_2(x_1)\phi_2(x_2)} : \phi_3(x_1)\phi_3(x_2) : + \overbrace{\phi_2(x_1)\phi_2(x_2)}\overbrace{\phi_1(x_1)\phi_1(x_2)} : \phi_3(x_1)\phi_3(x_2) :$$

$$+ \overbrace{\phi_3(x_1)\phi_3(x_2)}\overbrace{\phi_1(x_1)\phi_1(x_2)} : \phi_2(x_1)\phi_2(x_2) :$$

$$+ \overbrace{\phi_1(x_1)\phi_1(x_2)}\overbrace{\phi_2(x_1)\phi_2(x_2)}\overbrace{\phi_3(x_1)\phi_3(x_2)}$$

3 Propagators

Evaluate the function

$$\langle 0|\phi(x)\phi(y)|0\rangle = D(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-\iota p\cdot(x-y)}$$

for $(x-y)$ spacelike and $(x-y)^2 = -r^2$ explicitly in terms of Bessel functions.

Sol. Since $(x-y)$ is spacelike

$$(x-y)^\mu = (0, \vec{x}-\vec{y})$$

$$p^\mu = (E_p, \vec{p})$$

with $E_p = \sqrt{m^2 + \vec{p}^2}$. Hence

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\sqrt{m^2+\vec{p}^2}} e^{\iota\vec{p}\cdot\vec{x}-\vec{y})}$$

$$= \int_{\psi=0}^{2\pi} \int_{\cos\theta=-1}^{1} \int_{p=0}^{\infty} \frac{1}{(2\pi)^3} \frac{1}{2\sqrt{m^2+\vec{p}^2}} e^{\iota pr\cos\theta} dp d(\cos\theta) d\psi$$

$$= -\frac{\iota}{2(2\pi)^2 r} \int_{-\infty}^{\infty} \frac{p e^{\iota pr}}{\sqrt{m^2+p^2}} dp = \frac{m}{4\pi^2 r} K_1(mr)$$

4 Creation and Annihilation Operators

The action for a complex scalar field satisfying the Klein-Gordon Equation is given by

$$S = \int d^4x (\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi).$$

The corresponding Hamiltonian is given by

$$H = \int d^3x (\pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi).$$

Diagonalize H by introducing creation and annihilation operators. Show that the theory contains 2 particles of mass m.

Sol. Since ϕ is complex, it's real and imaginary parts are both solutions to the Klein-Gordon equation. So we can treat ϕ and ϕ^* independently instead of using the real and imaginary parts of ϕ. So we can write

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (a_p e^{-\iota\vec{p}\cdot\vec{x}} + b_p^\dagger e^{\iota\vec{p}\cdot\vec{x}})$$

$$\phi^*(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (b_p e^{-\iota\vec{p}\cdot\vec{x}} + a_p^\dagger e^{\iota\vec{p}\cdot\vec{x}}).$$

Conjugate momentum for ϕ and ϕ^* are

$$\pi = \frac{\partial L}{\partial\dot{\phi}} = \dot{\phi}^*$$

and

$$\tilde{\pi} = \frac{\partial L}{\partial\dot{\phi}^*} = \dot{\phi} = \pi^*$$

respectively. Substituting the partial time derivatives of ϕ and ϕ^* in the Hamiltonian we get

$$H = \int d^3x E_p (a_p^\dagger a_p + b_p^\dagger b_p).$$

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