Conformal Field Theory Approaches to Condensed Matter Theory

Submitted in partial fulfillment of the requirements for the degree of
Master of Technology in Nanoscience, and Bachelor of Technology in Engineering Physics

Dual Degree Project Report

Phase II

by

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INDIAN INSTITUTE OF TECHNOLOGY BOMBAY
(2018-19)

Dissertation Approval Certificate

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Abstract

In this report we have outlined the basics of XXZ-spin chain model. Upon bosonization of XXZ-spin chain model, we find that in $-1 < \Delta \leq 1$, it can be described by free bosonic CFT. We show how to calculate the partition function and expectation value of energy-momentum tensor on a n-copy Riemann surface with N branch cuts, $R_{n,N}$ using branch point twist fields. In particular, we notice that the partition on $R_{2,2}$ is proportional to the Euclidean four-point function of twist fields. We find the four-point function for Free-bosonic CFT and show that it is equivalent to four-point function of spin flip operator for XX-model. Finally we calculate the infinite temperature Out-of-Time-Order-Correlator (OTOC), and show that it decays as a power law for small times.

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1 Introduction

Field theories which remain invariant under conformal transformations are called Confomral Field Theories (CFTs). Conformal transformations are the transformations which leave the angles invariant, i.e., scale transformations, rotations, translations and special conformal transformations(inversion followed by translation and then again inversion). Such transformations are possible when the field theory has no dimensionful scales. Hence CFTs can be found in the continuum description of second order phase transitions and at the fixed points of renormalization group flow. CFTs are also found in string theories as the two-dimensional field theory living on the world-volume of a string which moves in some background space–time with dynamics governed by non-linear sigma model with vanishing β-functional [1].

With so many symmetries present, the system becomes very constrained and it is possible to study the system by exploiting its symmetries without even stating the classical action and quantizing it, the so-called bootstrap approach. This approach is specially useful in two dimensions where the number of independent infinitesimal conformal transformations is infinity. In the following, will try to outline some general properties of CFT in 2-dimensions which will be useful to us when we analyze the free bosonic CFT and make its connection with the XXZ model.

1.1 General properties of CFTs

Conformal transformations are transformations which preserve angle between any two lines. Hence, under a conformal transformation ϕ, the metric transforms as

$$g'_{\rho\sigma}(x')\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x) g_{\mu\nu}(x) \tag{1}$$

where $x' = \phi(x)$ and $\Lambda(x)$ is a positive function called scale factor. If the manifold on which which x and x' live are the same, then $g' = g$. In flat space, the metric is given by Minkowski metric $\eta_{\mu\nu} = diag(-1, 1, ...1)$ with $\Lambda(x) = 1$. Consider an infinitesimal transformation,

$$x'^{\rho} = x^{\rho} + \epsilon^{\rho}(x) + \mathcal{O}(\epsilon^2) \tag{2}$$

with $\epsilon << 1$. Substituting this in the L.H.S. of (1), we find that

$$\eta_{\rho\sigma}(x')\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \eta_{\mu\nu} + \left(\frac{\partial \epsilon_{\mu}}{\partial x^{\nu}} + \frac{\partial \epsilon_{\nu}}{\partial x^{\mu}}\right) + \mathcal{O}(\epsilon^2) \tag{3}$$

Comparing this with the R.H.S. of (1), we find that the second term in (3) must be proportional to metric. Taking the trace, we can find the proportionality constant such that,

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = \frac{2}{d}(\partial.\epsilon)\eta_{\mu\nu} \tag{4}$$

where d is the number of dimensions. This can be written as,

$$\Big(\eta_{\mu\nu}\Box + (d-2)\partial_{\mu}\partial_{\nu}\Big)(\partial.\epsilon) = 0 \tag{5}$$

As is clear from this, something special happens for $d = 2$. Contracting this with $\eta^{\mu\nu}$ gives

$$(d-1)\Box(\partial.\epsilon) = 0 \tag{6}$$

This means that ϵ_{μ} is at most quadratic in x^{ν}, i.e., $\epsilon_{\mu} = a_{\mu} + b_{\mu\nu}x^{\nu} + c_{\mu\nu\rho}x^{\nu}x^{\rho}$ with $a_{\mu}, b_{\mu\nu}, c_{\mu\nu\rho} << 1$. From this, we can find various transformations and their generators as given in the table below

Transformations		Generators
Translations	$x'^{\mu} = x^{\mu} + a^{\mu}$	$P_{\mu} = -\iota\partial_{\mu}$
Dilation	$x'^{\mu} = \alpha x^{\mu}$	$D = -\iota x^{\mu}\partial_{\mu}$
Rotation	$x'^{\mu} = M^{\mu}{}_{\nu} x^{\nu}$	$L_{\mu\nu} = \iota(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})$
SCT	$x'^{\mu} = \frac{x^{\mu}-(x.x)b^{\mu}}{1-2(b.x)+(b.b)(x.x)}$	$K_{\mu} = -\iota(2x_{\mu}x^{\nu}\partial_{\nu} - (x.x)\partial_{\mu})$

For $d \geq 3$, generators follow

$$[J_{mn}, J_{rs}] = \iota(\eta_{ms}J_{nr} + \eta_{nr}J_{ms} - \eta_{mr}J_{ns} - \eta_{ns}J_{mr}) \tag{7}$$

with $m, n = -1, 0, 1, ..., (d-1)$. Where we have defined

$$J_{\mu\nu} = L_{\mu\nu}, \qquad J_{-1,0} = D \qquad J_{-1,\mu} = \frac{1}{2}(P_{\mu} - K_{\mu}), \qquad J_{-1,\mu} = \frac{1}{2}(P_{\mu} - K_{\mu}) \tag{8}$$

For $\mathbb{R}^{p,q}$, this is the Lie algebra corresponding to $SO(p+1, q+1)$. For $d = 2$, (4) implies that

$$\partial_0\epsilon_0 = \partial_1\epsilon_1, \qquad \partial_0\epsilon_1 = -\partial_1\epsilon_0 \tag{9}$$

which can be identified as the Cauchy-Riemann equations where ϵ_0 and ϵ_1 are the real and imaginary part of a complex holomorphic function. If we redefine the complex variables, then

$$z = x^0 + \iota x^1 \qquad \epsilon = \epsilon^0 + \iota\epsilon^1 \qquad \partial_z = \frac{1}{2}(\partial_0 - \iota\partial_1) \tag{10}$$

and similarly for complex conjugates, then a holomorphic function, $f(z) = z + \epsilon(z)$ gives rise to an infinitesimal 2-D conformal transformation, $z \to f(z)$. Performing a Laurent expansion of $\epsilon(z)$ around $z = 0$, we get,

$$\begin{aligned} z' &= z + \epsilon(z) = z + \sum_{n\in\mathbb{Z}} \epsilon_n(-z^{n+1}) \\ \bar{z}' &= \bar{z} + \bar{\epsilon}(\bar{z}) = \bar{z} + \sum_{n\in\mathbb{Z}} \bar{\epsilon}_n(-\bar{z}^{n+1}) \end{aligned} \tag{11}$$

with generators,

$$l_n = -z^{n+1}\partial_z \qquad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \tag{12}$$

where l_n and $\bar{l}_n$ satisfy Witt algebra independently,

$$\begin{aligned} [l_m, l_n] &= (m-n)l_{m+n} \\ [\bar{l}_m, \bar{l}_n] &= (m-n)\bar{l}_{m+n} \\ [l_m, \bar{l}_n] &= 0 \end{aligned} \tag{13}$$

Since $n \in \mathbb{Z}$, this algebra is infinite dimensional. Global conformal transformations are generated by $\{l_{-1}, l_0, l_1\}$, and are of the form, $SL(2,\mathbb{C})/\mathbb{Z}_2$

$$z \to \frac{az+b}{cz+d} \;\; with \;\; a, b, c, d \in \mathbb{C} \;\; and \;\; ad - bc \neq 0 \tag{14}$$

The Witt algebra admits a central extension for infinitesimal conformal transformations, i.e., central extension $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}$ for Lie algebra $\mathfrak{g}$. The generators then satisfy Virasoro algebra.

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \tag{15}$$

From (15), it is clear that there is no non-trivial central extension for finite dimensional Lie algebras. In fact, from $m, n = -1, 0, 1$ we see that L_{-1} generates translationss, $L_0 \pm \bar{L}_0$ generates dilations and rotations respectively, and L_1 generates special conformal transformations.

Now consider Euclidean two-dimensional space $\mathbb{R}^2$ and identify it with $\mathbb{C}$ by $z = x^0 + \iota X^1$ and $\bar{z} = x^0 - \iota X^1$. The fields also depend on $(z, \bar{z})$. If the field only depends on z, i.e., $\phi(z)$, then it is called chiral or holomorphic and fields only dependent on $\bar{z}$ are called anti-chiral or anti-holomorphic. A field is said to have dimensions $(h, \bar{h})$ is under scaling transformation $z \to \lambda z$ it goes as

$$\phi(z, \bar{z}) \to \phi'(z, \bar{z}) = \lambda^h \bar{\lambda}^{\bar{h}} \phi(\lambda z, \bar{\lambda} \bar{z}) \tag{16}$$

Also, if under the transformation $z \to f(z)$, the field transforms as

$$\phi(z, \bar{z}) \to \phi'(z, \bar{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi(f(z), \bar{f}(\bar{z})) \tag{17}$$

then the field is said to be a primary field. If (17) is only satisfied for global conformal transformations, then the field is said to be quasi-primary. Energy momentum tensor is an example of quasi-primary fields. In a CFT, the fields can either be primary, quasi-primary (all primary fields are also quasi-primary) or a secondary field (descendent of a primary field). Under infinitesimal conformal transformations $z \to f(z) = z + \epsilon(z)$ for $\epsilon(z) << 1$, the primary field transforms as

$$\phi(z, \bar{z}) \to \phi(z, \bar{z}) + \delta\phi(z, \bar{z}) = \phi(z, \bar{z}) + \Big(h\partial_z \epsilon + \epsilon \partial_z + \bar{h}\partial_{\bar{z}}\bar{\epsilon} + \bar{\epsilon}\partial_{\bar{z}}\Big)\phi(z, \bar{z}) \tag{18}$$

Consider the conserved current, $j_\mu = T_{\mu\nu}\epsilon^\nu$, corresponding to conformal symmetry $x^\mu \to x^\mu + \epsilon^\mu(x)$. We can show that the energy-momentum tensor is traceless and the non-vanishing components are chiral, i.e., $T_{zz}(z, \bar{z}) = T(z)$ and $T_{\bar{z}\bar{z}}(z, \bar{z}) = \bar{T}(\bar{z})$. The corresponding conserved charge, Q is

$$Q = \int dx^1 j_0 = \frac{1}{2\pi\iota} \oint_C \Big(dz T(z)\epsilon(z) + d\bar{z}\bar{T}(\bar{z})\bar{\epsilon}(\bar{z})\Big) \tag{19}$$

Therefore,

$$\delta\phi(w, \bar{w}) = [Q, \phi(w, \bar{w})] = \frac{1}{2\pi\iota} \oint_C \Big[dz\epsilon(z) R\Big(T(z)\phi(w, \bar{w})\Big)\Big] + \frac{1}{2\pi\iota} \oint_C \Big[d\bar{z}\bar{\epsilon}(\bar{z}) R\Big(\bar{T}(\bar{z})\phi(w, \bar{w})\Big)\Big] \tag{20}$$

where $R\Big(A(z)B(w))\Big)$ represents radial ordering, i.e.,

$$\begin{aligned} R\Big(A(z)B(w))\Big) &= A(z)B(z) \quad for \quad |z| > |w| \\ &= B(w)A(z) \quad for \quad |w| > |z| \end{aligned} \tag{21}$$

Comparing (18) and (20), we find the operator product expansion between energy-momentum tensor and primary fields as

$$T(z)\phi(w, \bar{w}) = \frac{h}{(z-w)^2}\phi(w, \bar{w}) + \frac{1}{z-w}\partial_w \phi(w, \bar{w}) + ... \tag{22}$$

where we have assumed radial ordering. We can also find the operator product expansion of energy-momentum tensor with itself by expanding $T(z)$ in terms of its Laurent modes

$$T(z) = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n \tag{23}$$

where

$$L_n = \frac{1}{2\pi\iota} \oint dz z^{n+1} T(z) \tag{24}$$

and identifying $Q_n = -\epsilon_n L_n$. This gives

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + ... \tag{25}$$

Note that from (23) it is clear that chiral energy-momentum, $T(z)$ is a quasi-primary field of conformal dimensions $(h, \bar{h}) = (2, 0)$. This can be further verified by noticing that the general expansion

$$[L_m, \phi_n] = ((h-1)m - n)\phi_{m+n} \tag{26}$$

holds only for $m = -1, 0, 1$ if L_m is taken to be the generators of Virasoro algebra.

A very important property of energy-momentum tensor that we will be using in this report very frequently is its transformation under conformal transformations $w \to z$

$$T(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12}\{z, w\} \tag{27}$$

where $\{z, w\}$ is the Schwarzian derivative

$$\{z, w\} = \frac{z'''}{z'} - \frac{3}{2}\frac{z''^2}{z'^2} \tag{28}$$

Using the operator product expansion of primary fields of dimension Δ and using conformal symmetries on 2-dimensions, we can write the Conformal Ward Identities,

$$\langle T(z)\phi_1(w_1, \bar{w}_1)...\phi_1(w_N, \bar{w}_N)\rangle = \sum_{i=1}^{N}\left(\frac{\Delta_i}{(z-w_i)^2} + \frac{1}{(z-w_i)}\partial_{w_i}\right)\langle\phi_1(w_1, \bar{w}_1)...\phi_1(w_N, \bar{w}_N)\rangle \tag{29}$$

Using conformal symmetries, the two-point and three-point function of chiral quasi-primary fields can be fixed (for three-point function it is fixed upto some structure constant) to

$$\langle\phi_i(z)\phi_j(w)\rangle = \frac{d_{ij}\delta_{h_i,h_j}}{(z-w)^{2h_i}} \tag{30}$$

$$\langle\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = \frac{c_{123}}{z_{12}^{h_1+h_2-h_3} z_{23}^{-h_1+h_2+h_3} z_{13}^{h_1-h_2+h_3}} \tag{31}$$

For example,

$$\langle T(z)T(w)\rangle = \frac{c/2}{(z-w)^4} \tag{32}$$

For non-chiral fields $\phi_i(z, \bar{z})$, the $SL(2, \mathbb{C}/\mathbb{Z}_2) \times SL(2, \bar{\mathbb{C}}/\mathbb{Z}_2)$ symmetry determines the two-point function upto a normalization factor and three-point function upto a structure constant,

$$\begin{aligned}\langle\phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\rangle &= \frac{d_{12}\delta_{h_1,h_2}\delta_{\bar{h}_1,\bar{h}_2}}{z_{12}^{h_1+h2}\bar{z}_{12}^{\bar{h}_1+\bar{h}_2}} \\ \langle\phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\phi_3(z_3, \bar{z}_3)\rangle &= \frac{c_{123}}{z_{12}^{h_1+h_2-h_3} z_{23}^{-h_1+h_2+h_3} z_{13}^{h_1-h_2+h_3} \bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{23}^{-\bar{h}_1+\bar{h}_2+\bar{h}_3} \bar{z}_{13}^{\bar{h}_1-\bar{h}_2+\bar{h}_3}}\end{aligned} \tag{33}$$

Also, the four-point function can also depend only on the crossing ratios, $x = z_{12}z_{34}/z_{13}z_{24}$ and $\bar{x} = \bar{z}_{12}\bar{z}_{34}/\bar{z}_{13}\bar{z}_{24}$. Using the global symmetry, it is also possible to map any three points $\{z_1, z_2, z_3\}$ to $\{0, 1, \infty\}$ which will be useful when we study CFT on a torus.
Consider the Laurent expansion of chiral fields $\phi_i(z)$,

$$\phi_i(z) = \sum_m \phi_{(i)m} z^{-m-h_i} \tag{34}$$

Then the Laurent modes satisfy

$$
\begin{aligned}
[\phi_{(i)m}, \phi_{(j)n}] &= \sum_k C_{ij}^k p_{ijk}(m,n)\phi_{(k)m+n} + d_{ij}\delta_{m,-n}\binom{m+h_i-1}{2h_i-1} \\
p_{ijk}(m,n) &= \sum_{\substack{r,s\in\mathbb{Z}_0^+ \\ r+s=h_i+h_j-h_k-1}} C_{r,s}^{ijk}\binom{-m+h_i-1}{r}\binom{-n+h_j-1}{s} \\
C_{r,s}^{ijk} &= (-1)^r\frac{(2h_k-1)!}{(h_i+h_j-h_k-1)!}\prod_{t=0}^{s-1}(2h_i-2-r-t)\prod_{u=0}^{r-1}(2h_j-2-s-u)
\end{aligned}
\tag{35}
$$

For a chiral field $j_i(z)$ with conformal dimension $h=1$ (also called a current), where $i=1,...,N$, N is the number of quasi-primary currents in the theory, with Laurent expansion $j_i(z)=\sum_{n\in\mathbb{Z}} z^{-n-1}j_{(i)n}$, the Laurent modes satisfy

$$
[j_{(i)m}, j_{(j)n}] = \sum_l f^{ijl} j^l_{m+n} + km\delta^{ij}\delta_{m,-n} \tag{36}
$$

where we have rescaled d_{ij} to $k\delta_{ij}$ and identified $p_{111}=1$. f^{ijl} are called structure constants and are anti-symmetric in the first two indices, $f^{ijl}=-f^{jil}$, and k is called the level. This is a generalization of Kac-Moody algebra (which is not really relevant for this report).

Consider a field, $\phi(z,\bar{z})$, the Laurent mode expansion gives, $\phi(z,\bar{z})=\sum_{n,\bar{m}\in\mathbb{Z}} z^{-n-h}\bar{z}^{-\bar{m}-\bar{h}}\phi_{n,\bar{m}}$. Then ϕ_n with $n>-h$ are the annihilation operators and ϕ_n with $n\leq -h$ are the creation operators. The Normal ordering as usual means creation operators to the left. Normal ordered product of Laurent modes is

$$
N(\chi\phi)_n = \sum_{k>-h^\phi} \chi_{n-k}\phi_k + \sum_{k\leq -h^\phi} \phi_k\chi_{n-k} \tag{37}
$$

We then define the Verma module as any state $|\Phi\rangle$ which can be written as $L_{k_1}...L_{k_n}|0\rangle$ for $k_i\leq -2$. Each state in the Verma module can be written as

$$
|\Phi\rangle = \lim_{z\to 0} F(z)|0\rangle \qquad F\in\{T,\partial T,....N(...)\} \tag{38}
$$

For example, $T(z)=\lim_{z\to 0} L_{-2}|0\rangle$. Consider a primary field $\phi(z)$, with the heighest weight state, $|h\rangle$. Then the infinite set of descendent fields, formed by taking derivatives ∂^k and normal ordered products with T, is called the conformal family [1]

$$
[\phi(z)] = \{\phi,\partial\phi,\partial^2\phi,....,N(T\phi),....\} \equiv \{\hat{L}_{k_1}...\hat{L}_{k_n}\phi(z) : K_i\leq -1\} \tag{39}
$$

forming the tower of states

Field	State	Level
$\phi(z)$	$\phi_{-h}\|0\rangle = \|h\rangle$	0
$\partial\phi(z)$	$L_{-1}\phi_{-h}\|0\rangle$	1
$\partial^2\phi$	$L_{-1}L_{-1}\phi_{-h}\|0\rangle$	2
$N(T\phi)$	$L_{-2}\phi_{-h}\|0\rangle$	2
$\partial^3\phi$	$L_{-1}L_{-1}L_{-1}\phi_{-h}\|0\rangle$	3
$N(T\partial\phi)$	$L_{-2}L_{-1}\phi_{-h}\|0\rangle$	3
$N(\partial T\phi)$	$L_{-3}\phi_{-h}\|0\rangle$	3
...	...	...

At each level, n there are $P(n)$ different states, with

$$P(n) = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{N=0}^{\infty} P(N) q^N \tag{40}$$

The descendent fields, $\hat{L}_{-n}\phi$ can be defined as

$$\begin{aligned} T(z)\phi(w) &= \sum_{n\geq 0} (z-w)^{n-2} \hat{L}_{-n}\phi(w) \\ \hat{L}_{-n}\phi(w) &= \frac{1}{2\pi\iota} \oint dz \frac{1}{(z-w)^{n-1}} T(z)\phi(w) \end{aligned} \tag{41}$$

Then the correlators of descendents of primary fields, $\hat{L}_{-n}\phi$ can be defined by applying the differential operator $\mathcal{L}_{-n}$ on the primary field, ϕ as

$$\begin{aligned} \left\langle \hat{L}_{-n}\phi_1(w_1)...\phi_N(w_N) \right\rangle &= \mathcal{L}_{-n} \left\langle \phi_1(w_1)...\phi_N(w_N) \right\rangle \\ \mathcal{L}_{-n} &= \sum_{i=1}^{N} \left(\frac{(n-1)h_i}{(w_i-w)^n} - \frac{1}{(w_i-w)^{n-1}} \partial_{w_i} \right) \end{aligned} \tag{42}$$

1.2 Free and compactified bosonic CFT

So far, the general properties of CFT have been found solely based on the symmetry arguments without considering the Lagrangian. Now let us consider an example of CFT called free bosonic CFT, which we will later find is the CFT present in the XXZ model. Consider a real massless scalar field, $X(x^0, x^1)$ defined on a cylinder (a cylinder can be mapped to a complex plane by the transformation, $z = e^w = e^{x^0} e^{\iota x^1}$), with the compactification, $x^1 \simeq x^1 + 2\pi$. Consider the action S in Euclidean space as

$$S = \frac{1}{4\pi k} \int dx^0 dx^1 \Big((\partial_{x^0} X)^2 + (\partial_{x^1} X)^2 \Big) = \frac{1}{4\pi k} \int dz d\bar{z} \partial X . \bar{\partial} X \tag{43}$$

where k is some dimensionless constant (usually taken to be 1). Since there are no dimensionful parameters in this action, the theory is conformally invariant (if $X(z,\bar{z})$ has zero conformal dimensions). The equations of motion corresponding to this theory, $\partial_X S = 0$ are

$$\partial \bar{\partial} X(z, \bar{z}) = 0 \tag{44}$$

Hence we can define conformal currents $j(z)$ and $\bar{j}(\bar{z})$, which are chiral and anti-chiral respectively,as

$$\begin{aligned} j(z) &= \iota \partial X(z,\bar{z}) \\ \bar{j}(\bar{z}) &= \iota \bar{\partial} X(z,\bar{z}) \end{aligned} \tag{45}$$

From (44), we can find the two-point function for free bosonic theory to be

$$\langle X(z,\bar{z}) X(w,\bar{w}) \rangle = -k \log |z-w|^2 \tag{46}$$

which is not of the form, (30). Hence, $X(z,\bar{z})$ is not a quasi-primary field. Using (42), we can find the two-point function for the current fields, $j(z)$, and we find that

$$\langle j(z) j(w) \rangle = \frac{k}{(z-w)^2} \tag{47}$$

Hence $j(z)$ is quasi-primary field with $(h, \bar{h}) = (1, 0)$. The Laurent modes of current fields satisfy

$$[j_m, j_n] = k m \delta_{m+n,0} \tag{48}$$

Another primary field of free bosonic CFT are the so called Vertex operators, $V(z,\bar{z})$ with conformal dimensions $(h,\bar{h}) = (\frac{\alpha^2}{2}, \frac{\alpha^2}{2})$,

$$V_\alpha(z,\bar{z}) = :e^{\iota\alpha X(z,\bar{z})}: \tag{49}$$

where : ... : represents normal ordering. Note that since the action (43) is invariant under $X(z,\bar{z}) \to X(z,\bar{z}) + a$ for any constant a, the two-point correlator of vertex operators, $\langle V_\alpha V_\beta \rangle$ is non-zero only if $\alpha + \beta = 0$. Hence, α can be considered as the conserved charge of a vertex operator and the condition $\alpha + \beta = 0$, can be taken as charge conservation. The two point function is then given by

$$\langle V_{-\alpha}(z,\bar{z}) V_\alpha(w,\bar{w}) \rangle = \frac{1}{(z-w)^{\alpha^2} (\bar{z}-\bar{w})^{\alpha^2}} \tag{50}$$

Let us define the energy-momentum tensor as

$$T_{ab} = 4\pi k\gamma \frac{1}{\sqrt{|g|}} \frac{\delta S}{\delta g^{ab}} \tag{51}$$

where γ is some normalization constant. Then for action (43),

$$T_{zz} = \frac{1}{2k} N(\partial X \partial X) = \gamma N(jj), \qquad T_{z\bar{z}} = T_{\bar{z}z} = 0, \qquad T_{\bar{z}\bar{z}} = \frac{1}{2k} N(\bar{\partial} X \bar{\partial} X) \tag{52}$$

For the chiral part, (24) becomes,

$$L_n = \frac{1}{2k} \sum_{k>-1} j_{n-k} j_k + \frac{1}{2k} \sum_{k\leq -1} j_k j_{n-k} \tag{53}$$

we can then use this, along with (15) to calculate the central charge of the theory (then use $L_n |0\rangle = 0$), i.e.,

$$\langle 0|[L_2, L_{-2}]|0\rangle = \frac{c}{2} = \frac{1}{4k^2} \langle 0|j_1 j_1 j_{-1} j_{-1}|0\rangle = \frac{1}{2} \tag{54}$$

Therefore the free bosonic CFT has central charge, $c = 1$. Let us identify a special case of free bosonic theory with charge of the vertex operator as $\alpha = \pm\sqrt{2}$, for which the Vertex operator, $V_{\pm\sqrt{2}}(z,\bar{z})$ is a current $\big($has dimensions $(h,\bar{h}) = (1,1)\big)$. Consider,

$$V_{\pm\sqrt{2}}(z) = j^\pm(z) = :e^{\pm\iota\sqrt{2}X}: \tag{55}$$

Using the current algebra, (36) for $j^\pm$ and $j = \iota\partial X$, we find,

$$\begin{aligned} &[j_m, j_n] = m\delta_{m+n,0} && [j^\pm_m, j^\pm_n] = 0 \\ &[j_m, j^\pm_n] = \pm\sqrt{2} j^\pm_{m+n} && [j^+_m, j^-_n] = \sqrt{2} j_{m+n} + m\delta_{m+n,0} \end{aligned} \tag{56}$$

Redefining $j, j^\pm$ as

$$j^1 = \frac{1}{\sqrt{2}}(j^+ + j^-), \qquad j^2 = \frac{1}{\sqrt{2}\iota}(j^+ - j^-), \qquad j^3 = j \tag{57}$$

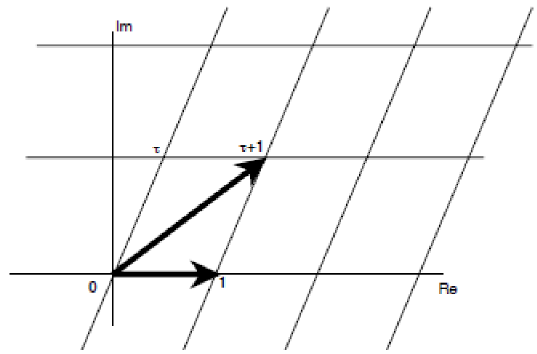
then (56), becomes

$$[j^i_m, j^j_n] = \iota\sqrt{2} \sum_k \epsilon^{ijk} j^k_{m+n} + m\delta^{ij}\delta_{m+n,0} \tag{58}$$

This represents $\mathfrak{su}(2)$ Kac-Moody algebra at level $k = 1$, which is identified with the isotropic ferromagnetic case in XXZ spin chain.

1.2.1 Free bosonic CFT on a torus

An important point to notice is that the choice of field, $X(z, \bar{z})$ is independent of the background metric. The only constraint is that the Manifold must be such that resulting theory can be conformally invariant. A torus is one such manifold with a compact surface and $g = 1$. Studying CFT on torus lead to constaints between the chiral and non-chiral sectors of the CFT [3]. It which can be constructed by taking a finite piece of complex cylinder and identifying the boundaries (or equivalently, taking a piece of the complex plane and identifying the boundaries.) A mathematically consistent definition of torus is complex plane modulo lattice, $\mathbb{C}/(\omega_1\mathbb{Z} + \omega_2\mathbb{Z})$, where $\omega_1, \omega_2 \in \mathbb{C}/\{0\}$ are the periods of the lattice. We can define the modular parameter as $\tau = \omega_2/\omega_1$



The lattice remains invariant under the modular group of torus,

$$\tau \to \frac{a\tau + b}{c\tau + d} \quad with \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})/\mathbb{Z}_2 \tag{59}$$

Specifically, the lattice remains invariant under the so called T-transformation $\big((w_1, w_2) = (1, \tau)$ to $(1, \tau + 1)\big)$, U-transformation $\big((w_1, w_2) = (1, \tau)$ to $(\tau + 1, \tau)\big)$ and S-transformation $\big((w_1, w_2) = (1, \tau)$ to $(-\tau, 1)\big)$ defined as

$$\begin{aligned} T \; &: \; \tau \to \tau + 1 \\ U \; &: \; \tau \to \frac{\tau}{\tau + 1} \\ S \; &: \; \tau \to -\frac{1}{\tau} \end{aligned} \tag{60}$$

T-invariance and S-invariance imply that the theory contains the maximal set of mutually local fields. Consider a mapping from a cylinder to a complex plane, $w \to z$ by

$$z = e^w = e^{x^0} e^{\iota x^1} \tag{61}$$

Since this maps time translations to dilations, we can write the Hamiltonian (generator of time translations) and momentum operator (generator of space translations) as

$$H_{cyl.} = (L_{cyl.})_0 + (\bar{L}_{cyl.})_0 \qquad\qquad P_{cyl.} = \iota((L_{cyl.})_0 - (\bar{L}_{cyl.})_0) \tag{62}$$

From (27), the energy-momentum operator transforms as

$$T_{cyl.}(w) = z^2 T(z) - \frac{c}{24} \qquad or \qquad (L_{cyl.})_0 = L_0 - \frac{c}{24} \tag{63}$$

The partition function on torus can then be written as

$$\mathcal{Z}(\tau,\bar{\tau}) = Tr\left(q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}\right) \tag{64}$$

where $q = e^{2\pi\iota\tau}$. Using $j(z) = \iota\partial X(z)$, we can write the Laurent mode as

$$L_0 = \frac{1}{2}j_0j_0 + \sum_{k\geq 1} j_{-k}j_k \tag{65}$$

Therefore, the partition function for a single free boson is of the form,

$$\mathcal{Z}_0(\tau,\bar{\tau}) = \frac{1}{|\eta(\tau)|^2} \tag{66}$$

where η is the Dedekind η-function given by

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n) \tag{67}$$

For the free boson compactified on a circle of radius R, where R^2 has to be even integer.

$$X(z,\bar{z}) \sim X(z,\bar{z}) + 2\pi Rn, \qquad n\in\mathbb{Z} \tag{68}$$

In terms of Laurent modes of the current, this means that

$$j_0 - \bar{j}_0 = Rn \qquad n\in\mathbb{Z} \tag{69}$$

Remember that for free boson, $j_0 = \bar{j}_0$. Since only one term in (65) changes, the partition function on compactified circle is given by

$$\begin{aligned}\mathcal{Z}_{cic.}(\tau,\bar{\tau}) &= \mathcal{Z}_0(\tau,\bar{\tau}).Tr(q^{\frac{1}{2}j_0^2}\bar{q}^{\frac{1}{2}\bar{j}_0^2}) \\ &= \frac{1}{|\eta(\tau)|^2}\sum_{e,n\in\mathbb{Z}} q^{\frac{1}{2}\left(\frac{e}{R}+\frac{Rn}{2}\right)^2}\bar{q}^{\frac{1}{2}\left(\frac{e}{R}-\frac{Rn}{2}\right)^2}\end{aligned} \tag{70}$$

In string theory, states with $n \neq 0$ are called winding states because they correspond to strings winding n times around the circle given by $X(z,\bar{z})$. States with $e \neq 0$ are called momentum or Kaluza–Klein states because, the centre of mass momentum ($\frac{j_0-\bar{j}_0}{2}$) of the string is quantised in a compact space [1].

The partition function, (70) is invariant under the symmetry $R \to 2/R$ along with $n \to m$, also known as the T-duality. $R = \sqrt{2}$ corresponds to the self-dual radius. The vertex operators for compactified bosons have, $\alpha = m/R$ for $m \in \mathbb{Z}$. Let us define character of an irreducible representation, $|h_i\rangle$, with Hilbert space $\mathcal{H}_i$ and highest weight, h_i as

$$\chi_i(\tau) \;:= Tr_{\mathcal{H}_i}\left(q^{L_0-\frac{c}{24}}\right) = \frac{\sum_n q^{\frac{1}{2}\left(\frac{m}{R}+\frac{Rn}{2}\right)^2}}{\eta(\tau)} \tag{71}$$

1.3 Orbifolds and Twist fields

Another choice for space apart from torus where conformal field theories make sense are orbifolds, manifolds with singularities. Orbifolds are defined as follows: Consider a manifold with a discrete symmetry. Define a new "orbifold" by considering the points related to each other by this symmetry as identical. Fixed points of the symmetry introduces conical singularities [2]. Therefore, it is necessary that all states obey the discrete symmetry of the theory (states which do not respect the discrete symmetry have to be projected out). Also, since some points are identified one can relax the boundary conditions of the boson. Instead of $X(x^1+2\pi) \sim X(x^1)$, or (68), we can have

$$X(x^1+2\pi) = GX(x^1) + 2\pi Rn \tag{72}$$

where G is some matrix representing the symmetry of the theory. This leads to an additional set of local fields for the theory called twist fields (in order to maintain the modular invariance of the theory, which is broken due to removal of some fields. Remember that modular invariant theories have a maximal set of mutually local fields). Twist fields are therefore non-local with respect to the fields that are removed. Consider for example the reflection symmetry of $\mathbb{Z}_2$ symmetry. To remove operators like ∂X, we introduce twist fields

$$\partial X(z)\sigma(w,\bar{w}) = (z-w)^{\Delta h}\sigma'(w,\bar{w}) + ... \tag{73}$$

This leads to branch cuts in the new "manifold". If we write the free bosonic theory in terms of an intermediate partially modular invariant theory, we can have both ∂X and σ to co-exist. The lowest dimension twist fields have dimensions $(h,\bar{h}) = (\frac{1}{16},\frac{1}{16})$. However, even though modular invariant theories on an orbifold do not contain ∂X, they do contain symmetric operators like

$$e^{\iota nR\phi} + e^{-\iota nR\phi}, \qquad n>0 \tag{74}$$

Hence, the lowest conformal weight of an operator in this theory is $R^2/2$. Due to the T-duality, the orbifold of $R=\sqrt{2}$ theory and the compactified boson with $R=2\sqrt{2}$, describe the same theory because at $R=\sqrt{2}$, the orbifold theory has spin-1 current.

Consider a lattice quantum field theory in $1+1$ dimensions, with lattice sites denoted by x and lattice spacing x. Let the complete set of eigenstates be denoted by $|\{\phi_x\}\rangle$. For a bosonic lattice field theory, these will be the fundamental bosonic fields of the theory; for a spin model some particular component of the local spin [5]. The density matrix ρ at inverse temperature β is

$$\rho(\{\phi_x\}|\{\phi'_{x'}\}) = \langle \otimes |\{\phi_x\}\rangle |\rho| \otimes |\{\phi'_{x'}\}\rangle\rangle = Z^{-1}\int [d\phi(y,\tau)] \prod_{x'} \delta(\phi(y,0)-\phi'_{x'}) \prod_{x} \delta(\phi(y,\beta)-\phi_x) e^{-S_E} \tag{75}$$

where $S_E = \int_0^\beta L d\tau$ is the Euclidean action. For $Tr\rho = 1$, we identify $\{\phi_x\}(0) = \{\phi'_{x'}\}(\beta)$. To calculate reduced density matrix ρ_A, where A consists of disjoint intervals $(u_1,v_1),...,(u_N,v_N)$, we only identify $\{\phi_x\}(0) = \{\phi'_{x'}\}(\beta)$ for x not in A. This leads to branch cuts along $\tau = 0$ (or $\tau = \beta$) for each interval (u_i,v_i). These branch cuts can then be identified with branch point twist fields. These branch cuts can be "removed" by making n copies of the Riemann surface, $R_{n,N}$. Twist fields then correspond to the boundary conditions, such that,

$$\begin{aligned} \tau_{n,N}\phi_i(x,0^+) &= \phi_{i+1}(x,0^-), \qquad x\in[u_j,v_j] \quad i=1,...,n; \quad j=1,...,N \\ \tilde{\tau}_{n,N}\phi_{i+1}(x,0^-) &= \phi_i(x,0^+) \end{aligned} \tag{76}$$

Twist fields also obey [7]

$$\langle O(z,\tau)\rangle_{\mathcal{L},R_{n,N}} = \frac{\langle O(z)\tau_N(w_1,\bar{w}_1)\tilde{\tau}_N(w_1,\bar{w}_1)...\tau_N(w_N,\bar{w}_N)\tilde{\tau}_N(w_N,\bar{w}_N)\rangle_{\mathcal{L}^{(n)},C}}{\langle \tau_N(w_1,\bar{w}_1)\tilde{\tau}_N(w_1,\bar{w}_1)...\tau_N(w_N,\bar{w}_N)\tilde{\tau}_N(w_N,\bar{w}_N)\rangle_{\mathcal{L}^{(n)},C}} \tag{77}$$

where w_i are the branch points. The partition function on $R_{n,N}$ is

$$\mathcal{Z}_{R_{n,N}} \propto \langle \tau_N(w_1,\bar{w}_1)\tilde{\tau}_N(w_1,\bar{w}_1)...\tau_N(w_N,\bar{w}_N)\tilde{\tau}_N(w_N,\bar{w}_N)\rangle_{\mathcal{L}^{(n)},C} \tag{78}$$

For example: Consider $N=1$, with no boundary conditions. Let the complex coordinate be $w = x+\iota\tau$ and $\bar{w} = x-\iota\tau$ where x lies in the interval $[u,v]$ of length $l = |u-v|$ in an infinitely long 1D quantum system, at zero temperature.

Consider a mapping $w \to z = ((w-u)/(w-v))^{1/n}$ which maps n-sheeted Riemann surface with 1 branch cut ($R_{n,1}$ where the branch points were at u, v) to the complex plane, z. Using this conformal mapping on equation (27) and taking thermal expectation values, with $\langle T(z)\rangle_C = 0$ due to translation and rotational symmetry of complex plane,

$$\langle T(w)\rangle_{R_{n,1}} = \frac{c}{12}\{z,w\} = \frac{c(1-1/n^2)}{24}\frac{(u-v)^2}{(w-u)^2(w-v)^2} \tag{79}$$

Using equation (29),

$$\langle T(w)\tau_n(u,0)\tilde{\tau}_n(v,0)\rangle_{\mathcal{L}^{(n)},C} = \left(\frac{\Delta_{\tau_n}}{(z-u)^2} + \frac{1}{(z-u)}\partial_u + \frac{\Delta_{\tilde{\tau}_n}}{(z-v)^2} + \frac{1}{(z-v)}\partial_v\right)\langle\tau_n(u,0)\tilde{\tau}_n(v,0)\rangle \tag{80}$$

For twist fields, scaling dimension $\Delta_{\tau_n} = \Delta_{\tilde{\tau}_n} = \Delta_n$. Also, from the symmetries of CFT (translational, rotational and scale invariance), the general form of two point function is given by

$$\langle\tau_n(u,0)\tilde{\tau}_n(v,0)\rangle = |u-v|^{-2\Delta_n} \tag{81}$$

If we substitute (81) in (80), and write it in the form of RHS of equation (77), we can equate it to equation (79) to get $\Delta_n = c/12(n-1/n)$ and and hence completely identify the two point function of twist fields. For $n = 2$, $\Delta_{\tau_2} = \Delta_{\tilde{\tau}_2} = c/8$. If we consider the underlying CFT to be free bosonic CFT with $c = 1$, then this scaling dimension and spin correspond to vertex operator with $\alpha = 1/2$.

1.4 Constrains on 4-point function using similar approach

Note that $R_{n,N}$ has a genus, $g = (n-1)(N-1)$, and is homeomorphic to a compact surface with g points removed. Hence, similar calculations cannot be done for any general n (because it cannot be mapped to a complex plane which has $g = 0$).

Here we are trying to use $n = 2$ and $N = 2$ (or $g = 1$), to find the four point function of twist fields. This can be relateed to the partition function on a torus. On a torus, the partition function is related to the energy-momentum tensor by [10]

$$\langle T\rangle_{torus} = 2\iota\pi\partial_\tau \ln \mathcal{Z}_{torus} \tag{82}$$

In fact, we can write the Ward identities on torus as

$$\begin{aligned}\langle T(z)\phi_1(w_1,\bar{w}_1)...\phi_1(w_N,\bar{w}_N)\rangle - \langle T\rangle\langle\phi_1(w_1,\bar{w}_1)...\phi_1(w_N,\bar{w}_N)\rangle \\ = \sum_{i=1}^{N}([h_i(\wp(w-w_i) + 2\eta_1) + (\xi(w-w_i) + 2\eta_1 w_i)\partial_{w_i}] \\ +2\iota\pi\partial_\tau)\langle\phi_1(w_1,\bar{w}_1)...\phi_1(w_N,\bar{w}_N)\rangle\end{aligned} \tag{83}$$

where Zeta-function, ξ and the Weiestrass $\wp$ are defined by

$$\begin{aligned}\xi(z) &= \frac{\partial_z\theta(z|\tau)}{\theta_1(z|\tau)} + 2\eta_1 z \\ \wp(z) &= -\partial_z\xi(z) \\ \eta_1 = \xi(\frac{1}{2}) &= -\frac{1}{6}\frac{\partial_z^3\theta(0|\tau)}{\partial_z\theta(0|\tau)}\end{aligned} \tag{84}$$

where θ_1 is the Jacobi theta function. For their full definitions and modular properties, see Appendix. It is important to note that $\wp$ defines a holomorphic map from $\mathbb{C}/L$ to $\mathbb{C} \cup \{\infty\}$, with branch points at $e_1, e_2, e_3, \infty = \wp(w_1/2), \wp(w_2/2), \wp((w_1 + w_2)/2), \infty$. Weiestrass $\wp$ function also satisfy,

$$\wp'(u)^2 = 4\wp^3(u) - g_2\wp(u) - g_3 \tag{85}$$

where $g_2 = 60G_4, g_3 = 140G_6$, and

$$G_r = G_r(L) = \sum_{w\in L} w^{-r} \tag{86}$$

Hence, e_1, e_2, e_3 are the roots of $w^2 = 4z^3 - g_2z - g_3$. And $u \to (z(u), w(u)) = (\wp(u), \wp'(u))$ defines a map from $\mathbb{C}/L\backslash\{0\} \to \mathbb{C}^2$. From this we can recover the periods and modular parameters of the torus by

$$w_1 = 2 * \int_{e_2}^{e_3} \frac{dz}{w(z)}, \qquad w_2 = 2 * \int_{e_1}^{e_3} \frac{dz}{w(z)} \tag{87}$$

This, together with (27) can be used to show that [8]

$$\langle \tau_2(0)\tau_2(x)\tau_2(1)\tau_2(\infty)\rangle = |2^8x(1-x)|^{-c/12}\mathcal{Z}(\tau, \bar{\tau}) \tag{88}$$

where x is the cross ratio

$$x = \frac{(u_1 - v_1)(u_2 - v_2)}{(u_1 - u_1)(v_1 - v_2)} \tag{89}$$

at $u_1, v_1, u_2, V_2 = 0, x, 1, \infty$. Note that the partition function on $R_{n,N}$ depends not only on the conformal structure of the manifold, but also on the conformal anomaly factor (metric and central charge of the theory) [9],

$$\mathcal{Z}_{R_{n,N}} = e^{cS_{anomaly}} \langle \tau_N(w_1, \bar{w}_1)\tilde{\tau}_N(w_1, \bar{w}_1)...\tau_N(w_N, \bar{w}_N)\tilde{\tau}_N(w_N, \bar{w}_N)\rangle \tag{90}$$

Hence, we find that

$$\langle \tau_2(u_1)\tau_2(v_1)\tau_2(u_2)\tau_2(v_2)\rangle = \frac{\mathcal{Z}_{R_{2,2}}}{c_n^2} = \left(\frac{|u_1 - u_2||v_1 - v_2|}{|u_1 - v_1||u_2 - v_2||u_1 - v_2||u_2 - v_1|}\right)\mathcal{F}_n(x) \tag{91}$$

For Free bosonic theory, compactified on a circle of radius R, [7] $\mathcal{F}_2(x)$ has been calculated to be

$$\mathcal{F}_2(x) = \frac{\theta_3(\eta\tau)\theta_3(\tau/\eta)}{[\theta_3(\tau)]^2} \tag{92}$$

where τ is purely imaginary. $\theta_v(\tau)$ are the Jacobi theta functions and η is related to the Luttinger parameter K, which appears again in equation (134), by

$$\eta = \frac{1}{2K} \tag{93}$$

The crossratio, x can be written in terms of modular parameter as

$$x = \left(\frac{\theta_2(\tau)}{\theta_3(\tau)}\right)^2, \qquad or \qquad \tau(x) = \iota\frac{F(1-x)}{F(x)} \tag{94}$$

where $F(x) = {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)$ are the Hypergeometric functions.

A similar analysis can be done for $R_{n,N}$ because of the uniformization theorm, which states that Every connected Riemann surface R is isomorphic to one of the following [12]:

- $\mathbb{C} \cup \infty$
- $\mathbb{C}$, $\mathbb{C}^*$ or $\mathbb{C}/L$

- Δ/Γ where Δ is the open unit disc and $\Gamma \subset$ PSU(1,1) is a discrete group of automorphisms acting freely.

A map which does this was provided in [13] as

$$w^n = u(z)v(z)^{n-1}, \qquad u(z) = \prod_{j=0}^{m}(z - z_{2j+1}), \qquad v(z) = \prod_{j=1}^{m}(z - z_{2j}) \qquad n > 1 \tag{95}$$

where z_i are the branch points. For $n = 2$, we verify that it is of the form, $w^2 = 4z^3 - g_2 z - g_3$.

2 Spin-1/2 XXZ chain

Consider the Hamiltonian given by

$$H = -J\sum_{n=1}^{N}[S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + \Delta S_n^z S_{n+1}^z] - 2h\sum_{n=1}^{N} S_n^z \tag{96}$$

where $S_j^\alpha = \frac{1}{2}\sigma_j^\alpha$ with $\alpha = 1, 2, 3$ are the spin-1/2 operators (σ_j^α are Pauli matrices) on the j-th lattice site of a spin chain of N-sites with periodic boundary conditions, $S_{1+N}^\alpha = S_1^\alpha$. The external magnetic field h is directed along the z-direction.The spin chain system is generally referred to as XXZ-spin chain or Heisenberg-Ising model.

In this report we will use the convention $J = 1$ and $h = 0$ (Note that, depending on whether $J > 0$ or $J < 0$, there will be ferromagnetic or antiferromagnatic order in x-y plane. They are related by a π-rotation about every other spin and then transforming $\Delta \to -\Delta$.) Hence, for us $\Delta > 0$ corresponds to ferromagnet while $\Delta < 0$ corresponds to antiferromagnet (We will see later that $-1 \le \Delta \le 1$ corresponds to paramagnetic case. Hence the above statement is only true for $|\Delta| > 1$.)

We will note certain general properties of the XXZ model and then shift our focus to the paramagnetic regime (which is where the model has conformal symmetry).

- For an even N, the energy spectrum of the Hamiltonian $H(\Delta)$ and $H(-\Delta)$ are related by a reflection around $E = 0$.

- This Hamiltonian commutes with total spin along the z-axis,

$$[H, \sum_{n=0}^{N} S_n^z] = 0 \tag{97}$$

- For $\Delta > 1$, we have gapped ferromagnet with low energy excitations corresponding to individual magnons or their bound states. $\Delta = 1$ is the isotropic ferromagnet (also called XXX chain or Heisenberg chain) and has a gapless spectrum.

- The spin chain is critical in the region $|\Delta| \le 1$, i.e. the ground state has zero magnetization and corresponds to spin vacuum. The low energy states are thus spinons (for certain interval of $\Delta > 0$, $\Delta \in (cos(\pi/(m+1)), 1)$ bound states of m magnons also become stable.) $\Delta = 0$ corresponds to the non-interacting isotropic limit of XY chain (also called XX chain.)

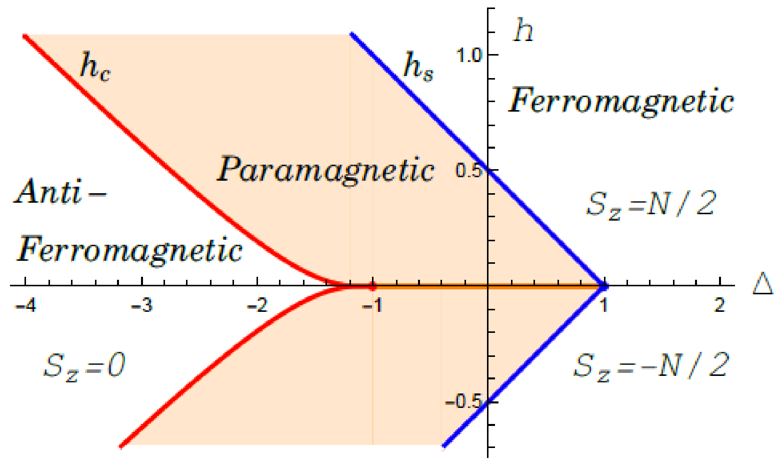


Figure 1: Phase diagram of XXZ chain. [14]

- For $\Delta < -1$, we have gapped antiferromagnet. At zero magnetization, the ground state is gapped (Note that, as evident from figure(1), this is only true upto a critical value of magnetic field, h_c, after which the model becomes gapless and the ground state becomes paramagnetic. At an even higher magnetic field h_s, the ground state becomes completely polarized and the phase becomes gapped ferromagnetic.) $\Delta = -1$ is the Antiferromagnetic Heisenberg state.

2.1 Jordan-Wigner transformation and bosonization

Instead of working with spin operators, it is more convenient to map the spins into either fermionic or bosonic operators. This is because the spin operators obey different commutation relations on-site and between the sites (on each site they span a finite dimensional antisymmetric Fock space (a property associated with fermions), while between the sites they obey bosonic commutation relations)

Jordan-Wigner transformation maps these spin operators to bosonic operators (however note that in the process the interactions become non-local). In terms of spinless fermions,

$$
\begin{aligned}
S^z_j &= \psi^\dagger_j \psi_j - \frac{1}{2}, \qquad S^+_j = -\frac{1}{2}\exp\left\{\iota\pi \sum_{l<j} \psi^\dagger_l \psi_l\right\} = \prod_{l=1}^{j-1}\left(\psi^\dagger_l \psi_l - \frac{1}{2}\right)\psi_j \\
S^-_j &= -\frac{\psi^\dagger_j}{2}\exp\left\{-\iota\pi \sum_{l<j} \psi^\dagger_l \psi_l\right\} = \prod_{l=1}^{j-1}\left(\psi^\dagger_l \psi_l - \frac{1}{2}\right)\psi^\dagger_j
\end{aligned}
\tag{98}
$$

where $S^\pm = (S^x \pm \iota S^y)/2$ are the fermionic annihilation and creation operators, which satisfy, $\{\psi_j, \psi^\dagger_k\} = \delta_{jk}$ and $\{\psi_j, \psi_k\} = 0$. Therefore on each site, up-spin is mapped to an empty state, while down-spin is mapped to occupied state. The non-local part of this mapping (also called Jordan-Wigner string), counts the parity of overturned spins to the left of the site at which it is applied, and hence takes care of the the bosonic commutation relation between the sites.

Further defining, $S^z_j := \psi^\dagger_j \psi_j$, the Hamiltonian becomes

$$
H = -\sum_{n=1}^{N}\left[\frac{1}{2}(S^+_n S^+_{n+1} + S^-_n S^-_{n+1}) + \Delta S^z_n S^z_{n+1}\right]
\tag{99}
$$

We see that this system is very constrained because of the non-local effects (exciting individual particle causes all other particles to rearrange) which makes it possible to study the system in terms of density of particles, provided that the density field,

$$\rho(x) = \sum_j \delta(x - x_j) \tag{100}$$

is a smooth function (x_j is the position of j-th particle.) Equation (99) can be written as the sum of free fermionic Hamiltonian and an interaction term. We will study the bosonization of free fermionic Hamiltonian first.

2.1.1 Bosonization of free fermionic Hamiltonian

Consider the Hamiltonian,

$$\mathcal{H}_0 = -\frac{1}{2m}\Psi^\dagger(x)\partial_x^2\Psi(x) = \frac{k^2}{2m}\tilde{\Psi}^\dagger(k)\tilde{\Psi}(k) \tag{101}$$

We can define the left and right moving fields $\psi_\pm$ as

$$\psi_\pm = \frac{1}{\sqrt{2\pi}} : e^{\mp\iota\sqrt{4\pi}\phi_\pm(x)} : \tag{102}$$

where $\phi_\pm$ are the collective bosonic fields and $: O :$ represents the normal ordering. The $\sqrt{4\pi}$ is related to the compactification radius and is chosen such that the anticommutation relations of fermionic fields translates into appropriate commutation relations for bosonic fields. Hence the free fermionic Hamiltonian can be written as

$$\mathcal{H}_0 = -\frac{1}{2m}\Big(\psi_+^\dagger(\partial_x + \iota k_F)^2\psi_+ + \psi_-^\dagger(\partial_x - \iota k_F)^2\psi_+\Big) \tag{103}$$

For low energy excitations, we can expand around the left and right Fermi points linearly as

$$\mathcal{H}_0 \simeq -\frac{k_F}{2m}\big(\psi_+^\dagger\psi_+ + \psi_-^\dagger\psi_-\big) - \iota\frac{k_F}{m}\big(\psi_+^\dagger\partial_x\psi_+ - \psi_-^\dagger\partial_x\psi_-\big) + ... \tag{104}$$

Here the first term is like the Fermi energy, while the second term corresponds to excitations around Fermi points $\pm k_F$. By expanding $\Psi(x)$ around left and right Fermi point, we can find the chiral fields as

$$\psi_\pm(x) = \int_{\pm k>0}\frac{dk}{2\pi}e^{\iota(k\mp k_F)x}\tilde{\Psi}(k) \tag{105}$$

Hence,

$$\Psi(x) = e^{\iota k_F x}\psi_+(x) + e^{-\iota k_F}\psi_-(x) := \frac{e^{\iota k_F x}}{\sqrt{2\pi}}e^{-\iota\sqrt{4\pi}\phi_+(x)} + \frac{e^{-\iota k_F x}}{\sqrt{2\pi}}e^{\iota\sqrt{4\pi}\phi_-(x)} \tag{106}$$

Note that while calculating fermionic bilinears in terms of bosonic fields, we need to regularize coordinate space by discretizing because square of fields (appearing due to normal ordering of fields) is not defined. Also we need to add regulators to prevent momentum from becoming too large. This process is called point splitting. From (105), the generating function for chiral fermionic current is of the form,

$$j_n^\pm(x) = \psi_\pm^\dagger(x)\partial_x^n\psi_\pm(x) \tag{107}$$

Hence,

$$\rho_\pm = j_0^\pm = \psi_\pm^\dagger(x)\psi_\pm(x) = -\frac{1}{\sqrt{\pi}}\partial_x\phi_\pm(x), \tag{108}$$

$$j_1^\pm(x) = \psi_\pm^\dagger(x)\partial_x\psi_\pm(x) = \pm\iota(\partial_x\phi_\pm(x))^2 - \frac{1}{\sqrt{4\pi}}\partial_x^2\phi_\pm \tag{109}$$

Therefore we can write (104) as

$$\mathcal{H}_0 \simeq -\iota\frac{k_F}{m}[j_1^+ - j_1^-] + ... = \frac{k_F}{m}\Big[(\partial_x\phi_+)^2 + \partial_x\phi_-)^2\Big] + ... \tag{110}$$

where we have shifted the ground state by chemical potential (first term in 104.) Let us define the bosonic field, and its dual by

$$\phi(x) = \phi_+(x) + \phi_-(x) \qquad\qquad \theta(x) = \phi_+(x) - \phi_-(x) \tag{111}$$

Remember that $\Psi(x)$ and $\Psi^\dagger$ obey fermionic commutation relations. Then from (106),

$$[\phi(x), \theta(y)] = \iota\Theta_H(y-x) \tag{112}$$

where $\Theta_H(x)$ is the Heaviside step function. Therefore the derivative of $\theta(x)$ satisfies

$$[\phi(x), \partial_y\theta(y)] = \iota\delta(y-x) \tag{113}$$

We will identify $\Pi(x) = \partial_y\theta(y)$ as the conjugate of $\phi(x)$. Equation (110) will become

$$\mathcal{H}_0 = \frac{v_0}{2}\int \Big[(\Pi(x))^2 + (\partial_x\phi(x))^2\Big]dx \tag{114}$$

where $v_0 = k_F/m$ is sound velocity of the free system. We find that $\phi(x)$ is similar to the non-linear part of the Jordan-Wigner transformation in the sense that $\phi(x)$ counts the number of particles to the left of x, and its derivatives give the particle density.

A very important observation that can be made from equation (114) is that all the terms in it are marginal (remember in 1+1 D, $d_\phi = 0$, and terms with $d_O = 2$ are marginal.) Hence, any **interaction** once bosonized acts like an **irrelevant** term or a combination of $(\Pi(x))^2$ and $(\partial_x\phi(x))^2$, as long as it stays in the critical region. Under the RG flow, the Hamiltonian becomes,

$$\mathcal{H}_0 = \frac{v_s}{\pi}\int \Big[K(\Pi(x))^2 + \frac{1}{K}(\partial_x\phi(x))^2\Big]dx \tag{115}$$

where v_s is the renormalised Fermi velocity of the interacting system and K is the Luttinger parameter. Depending upon the value of K $\big(K \in (0,1),\ K = 1,\ K > 1\big)$, the system corresponds to repulsive fermions, free fermions or attractive fermions.

Even if we add relevant terms to the Hamiltonian (interactions which open a gap in the ground state,) usually sine or cosine terms in the field or its dual, we can extract the behaviour of the system using suitable Sine-Gordon model. This process of bosonization is very general and any one dimensional critical system can be bosonized. For spin-chains systems, we first perform a Jordan-Wigner transformation to map the spins to fermions and then bosonize those fermions. In perticular, we can now bosonize the XXZ Hamiltonian (99). At the half-filling, i.e., $k_F = \pi/2$, the magnetization is zero and the particle density is given by

$$\rho(x) = \rho_0 - \frac{1}{\sqrt{\pi}}\partial_x\phi(x) + \frac{1}{\pi}cos(\sqrt{4\pi}\phi(x) - x) \tag{116}$$

2.1.2 Bosonization of XXZ model

At half filling, from (116), we find that certain quantities can be written as a sum of a smooth and an oscillating components. For spin density, let us define

$$\begin{aligned} S^z(x) &= \rho(x) + (-1)^n M(x) \\ \rho(x) &= :\psi_+^\dagger(x)\psi_+(x): + :\psi_-^\dagger(x)\psi_-(x): = \frac{1}{\sqrt{\pi}}\partial_x\phi(x) \\ M(x) &= :\psi_+^\dagger(x)\psi_-(x): + :\psi_-^\dagger(x)\psi_+(x): = -\frac{1}{a\pi} :sin(\sqrt{4\pi}\phi(x)): \end{aligned} \tag{117}$$

Here a is the lattice spacing and we have redefined the chiral fields such that, $\psi_j = \sqrt{a}[(-\iota)^j\psi_+(x) + \iota^j\psi_-(x)]$. Writing (99) as $H = \mathcal{H}_0 + H_{int}$, then upto a linear approximation, $\mathcal{H}_0$ is given by (110) or (114), while the interaction term can be written as

$$H_{int} = v_0\Delta \int dx[: \rho(x)\rho(x+a) : -M(x)M(x+a)] \tag{118}$$

Hence, using (117) in H_{int}, and using (114) we can write the XXZ Hamiltonian as

$$H = \int dx \left\{ \frac{v_0}{2}\left[\Pi^2 + \left(1 + \frac{4\Delta}{\pi}\right)(\partial_x\phi)^2)\right] + \frac{v_0\Delta}{(\pi a)^2} : cos(\sqrt{16\pi}\phi) : \right\} \tag{119}$$

From the conformal dimensions of the cosine term, we see that for $|\Delta| < 1$, the term is irrelevant and the Hamiltonian is given by (115). At $\Delta = -1$, the cosine term (which originates from the so called Umklapp processes, where 2 particles are removed from a Fermi point and added at the other, corresponding to a transfer of $4k_f$ momentum.) turns relevant and the the system becomes gapped antiferromagnet. At $\Delta = 1$, the low energy excitations are magnons with quadratic dispersion and not spinons, and hence there is a Galilian symmetry which breaks the bosonization process.

In the following basic results of Bethe ansatz solutions of XXZ model are described. While the solutions are available for all phases of the spin chain system, we will only outline the paramagnetic phase. For general solutions, the reader is referred to [11]. Consider a parametrization in terms of $\tilde{\lambda}_j$, called rapidity (here the equations are given in terms of Orbach's redefinition of rapidity as $\lambda = \iota\gamma\tilde{\lambda}_j$) to parametrize quasi-momenta k_j as

$$e^{ik_j} = -\frac{\sinh\frac{1}{2}(\lambda_j - \iota\gamma)}{\sinh\frac{1}{2}(\lambda_j + \iota\gamma)} \qquad -\infty < \mathcal{R}e(\lambda_j) < \infty \tag{120}$$

or equivalently,

$$k_j = 2\arctan\left[\frac{\tanh\lambda_j/2}{\tan\gamma/2}\right] = f(\lambda_j \mid \gamma/2) \qquad -(\pi-\gamma) < \mathcal{R}e(k) < \pi - \gamma \tag{121}$$

where γ is defined as

$$\Delta = -\cos\gamma \qquad (0 < \gamma < \pi) \tag{122}$$

The phase function is,

$$-e^{\iota\theta(k,k')} = \frac{\sinh\frac{1}{2}(\lambda - \lambda' - 2\iota\gamma)}{\sinh\frac{1}{2}(\lambda - \lambda' + 2\iota\gamma)} \tag{123}$$

or

$$\theta(k,k') = 2\arctan\left[\frac{\tanh(\lambda-\lambda')/2}{\tan\gamma}\right] = f(\lambda - \lambda' \mid \gamma) \tag{124}$$

The Bethe equations are given by

$$N\theta(\lambda_j \mid \gamma/2) = 2\pi I_j(\lambda) + \sum_{\lambda'}\theta(\lambda_j - \lambda' \mid \gamma), \qquad j = 1, 2, ..., R \tag{125}$$

where $\{I_j\}$ are the integer/half-integer quantum numbers defining the state with R spins flipped at lattice sites $\{n_j\}$. To understand excited states, consider the spectral decomposition of the two point function,

$$\langle\phi(w_1,\bar{w}_1)\phi(w_2,\bar{w}_2)\rangle = \sum_Q |\langle 0|\phi(0,0)|Q\rangle|^2 e^{-(-\tau_1-\tau_2)(E_Q-E_0)+\iota(x_1-x_2)(P_Q-P_0)} \tag{126}$$

where the ground state E_0, is given by pure, real rapidities filling the Fermi sea. Comparing this with standard CFT on a cylinder results, we find

$$\begin{aligned} E_Q - E_0 &= \frac{2\pi v_s}{L}(\Delta^+ + \Delta^-) \\ P_Q - P_0 &= \frac{2\pi}{L}(\Delta^+ - \Delta^-) \end{aligned} \tag{127}$$

where the Fermi velocity (v_s) is the derivative of dressed energy ($\epsilon(\lambda)$) with respect to dressed momentum ($k(\lambda)$) at the Fermi point Λ,

$$v_s = \left.\frac{\partial\epsilon(\lambda)}{\partial k(\lambda)}\right|_{\lambda=\Lambda} = \frac{1}{2\pi\rho(\Lambda)}\left.\frac{\partial\epsilon(\lambda)}{\lambda}\right|_{\lambda=\Lambda} \tag{128}$$

Let us define the dressed charge function, $Z(\lambda)$ which has some interesting relations with other thermodynamic quantities as [15]

$$Z(\lambda) + \frac{1}{2\pi}\int_{-\lambda}^{\Lambda}\mathcal{K}(\lambda-\nu)Z(\nu)d\nu = 1 \tag{129}$$

where

$$\mathcal{K}(\lambda) = \frac{d}{d\lambda}\theta(\lambda|\gamma/2) \tag{130}$$

Using Bethe ansatz, we can find v_s, $E_Q - E_0$ and $P_Q - P_0$ for XXZ model,

$$v_s^{XXZ} = \frac{\pi\sin\gamma}{2\gamma} \tag{131}$$

$$\begin{aligned} E_Q - E_0 &= \frac{2\pi v_s}{L}\left[\left(\frac{\Delta N}{2\mathcal{Z}}\right)^2 + (\mathcal{Z}d)^2 + N^+ + N^-\right] \\ P_Q - P_0 &= 2k_f d + \frac{2\pi}{L}(N^+ - N^- + \Delta N d) \end{aligned} \tag{132}$$

where $\mathcal{Z} = Z(\Lambda) = Z(-\Lambda)$ is the value of dressed charge at the Fermi point, $N^\pm$ represents the change in quantum numbers when particles at the Fermi point $\pm\Lambda$ are boosted and ΔN is the number of particles added (or subtracted) to (from) the system and placed (removed) around the Fermi points. Comparing with CFT values, we find the the conformal dimensions of operators corresponding to excitations,

$$\Delta^\pm = \frac{1}{2}\left(\frac{\Delta N}{2\mathcal{Z}} \pm \mathcal{Z}d\right)^2 + N^\pm \tag{133}$$

In CFT language $N^\pm$ describes the level of the descendants and ΔN is a characteristic of the local field $\phi(x,t)$. The spin flip correlation function $\langle S^-(x,\tau)S^+(0,0)\rangle$ in the paramagnetic regime corresponds to $\Delta N = 1$ and $d = N^\pm = 0$ [14]. Hence, $\Delta_{S^\pm} = 1/8\mathcal{Z}^2$. The value of dressed charge at Fermi point is related to the Luttinger parameter, K by

$$K = \mathcal{Z}^2 \tag{134}$$

The Luttinger parameter, K for XXZ chain is

$$K^{XXZ} = \frac{\pi}{2(\pi-\gamma)} \tag{135}$$

Also, from finite size calculations in CFT,

$$E \simeq Le - c\frac{\pi}{6L}v_s \tag{136}$$

Sub (131) in (136) and comparing with the standard calculations for XXZ model using Bathe ansatz for finite size, we find that $c = 1$.

Hence, $\Delta = 0$ corresponds to free fermions, $K = 1$, $\Delta > 0$ corresponds to repulsive fermions, $K < 1$ and $\Delta < 0$ corresponds to attractive fermions $K > 1$. Therefore in the critical region, we can write

$$S^z(x) = \sqrt{\frac{K}{2\pi}}\partial_x\phi(x) - (-1)^n \frac{1}{a\pi} : sin(\sqrt{4\pi K}\phi(x)) : \tag{137}$$

and

$$S^{\pm}(x) = constant(-1)^n e^{\pm\iota\sqrt{\frac{\pi}{K}}\theta(x)} \tag{138}$$

3 OTOC Calculation for XXZ chain

Quantum chaos, unlike classical chaos cannot be described as extreme dependency of the system on its initial conditions. We have look for other characteristics to verify its manifestation. Quantum chaos has its manifestations in spectral irregularities and related phenomena observable in quantum systems with few degrees of freedom whose classical counterparts are dynamically nonintegrable [16]. In particular, it is observed that general integrable systems have Poisson distribution of energy level spacing, which implies the existence of many degenerate eigenenergies. On the other hand, the quantum chaotic systems have the WignerDyson distribution of energy level spacing. The WignerDyson distribution has two main features: zero probability at zero energy level spacing and a peak at a finite energy level spacing. The former feature means that there is a little degeneracy while the latter implies that there are a large number of energy level spacings around the peak value [17].

3.1 OTOC

Also, quantum chaos causes a quantum system to behave statistically in the sense that the time evolution of a dynamical variable approaches an equilibrium value and that this value is the one predicted by quantum statistical mechanics. Both of these properties seem to be satisfied by XXZ spin chain in the limit $N \to \infty$, however the correlations are quite long [18].

In the recent years, the focus has been brought back on Out-of-Time-Ordered-Correlators (OTOC) which are diagnostics of chaos [19]. Consider

$$c(t) = -\left\langle [W(t), V(0)]^2 \right\rangle \tag{139}$$

The commutator diagnoses the effect of perturbations by V on later measurements of W and vice versa [20], where $\langle . \rangle$ is the thermal expectation value, $Z^{-1}Tr(e^{-\beta H})$ at inverse temperature, $\beta = T^{-1}$. For large t, this decomposes as $2\langle VV \rangle \langle WW \rangle$. For a finite but large t_*, $C(t)$ becomes significant and this time is called scrambling time. For a shorter time scale, the exponential decay time, t_d, general time ordered correlators approach their asymptotic values. This time scale is called dissipation time. If there is a large hierarchy between these time scales, it was shown in [20], that chaos can develop no faster than the Einstein gravity result in thermal quantum systems with many degrees of freedom, i.e.,

$$\lambda_L \leq \frac{2\pi}{\beta} = 2\pi T \tag{140}$$

where λ_L is the Lyapunov exponent. Due to the singularities of correlation function at twice insertion, it is necessary to regularize (139). To do so, we can move the Euclidean commutator halfway across the thermal circle by

$$-Tr\big(y^2[W(t), V]y^2[W(t), V]\big) \tag{141}$$

where

$$y^4 = \frac{1}{Z}e^{-\beta H} \tag{142}$$

Instead of working with (141), we would like to work with the OTOC, which can be defined as

$$F(t) = Tr(yVyW(t)yVyW(t)) \tag{143}$$

which is related to (141) by

$$\begin{aligned} -Tr\big(y^2[W(t),V]y^2[W(t),V]\big) &= Tr(y^2VW(t)y^2W(t)V) + Tr(y^2W(t)Vy^2VW(t)) \\ &\quad -Tr(y^2VW(t)y^2VW(t)) - Tr(y^2W(t)Vy^2W(t)V) \end{aligned} \tag{144}$$

Where $Tr(y^2VW(t)y^2VW(t)) = F(t - \iota\beta/4)$ and $Tr(y^2W(t)Vy^2W(t)V) = F(t + \iota\beta/4)$. From this and our discussion on behaviour of (139) for different t, we see that, at large t, F should become small. F(t) can also be used to find the Lyapunov exponent by [22]

$$\begin{aligned} \langle X_1(\theta_1)X_3(\theta_3)X_2(\theta_2)X_4(\theta_4)\rangle &= \langle X_1(\theta_1)X_2(\theta_2)\rangle \langle X_3(\theta_3)X_4(\theta_4)\rangle \\ &\quad -e^{\iota\lambda_L(\pi-\theta_1-\theta_2+\theta_3+\theta_4)/2}\Gamma(\theta_1-\theta_2,\theta_3-\theta_4) \end{aligned} \tag{145}$$

Depending upon whether F(t) becomes small faster than power law (from 2-point function) or not, we might or might not be able to extract out the Lyapunov exponent from it.

3.2 Analytic continuation

Since we have found the Euclidean 4-point function for $c = 1$ free bosonic CFT, we would like to use that to extract the 4-point function and OTOC in Lorentzian theories. Assuming some bound on the growth of correlators, good Euclidean theories are always in one-to-one correspondence with good Lorentzian theories [23]

Due to presence of branch cuts and singularities, a single Euclidean correlator can correspond to multiple Lorentzian correlators. An easy way to get the desired Lorentzian correlator is to use the $\iota\epsilon$ prescription.

$$\langle O_1(t_1,x_1)O_2(t_2,x_2)...O_N(t_N,x_N)\rangle = \lim_{\epsilon_j\to 0}\langle O_1(t_1-\iota\epsilon_1,x_1)...O_N(t_N-\iota\epsilon_N,x_N)\rangle \tag{146}$$

where the limit is taken with $\epsilon_1 > \epsilon_2 > ... > \epsilon_N > 0$. This process is valid not only on Minkowski space but also on the Lorentzian cylinder [25]. Hence to calculate finite temperature OTOC, we can make the transformation $w \to w' = (\beta/2\pi)\log w$. Since for now we are interested in calculating OTOC for twist fields in free bosonic theory, we can use (17) with above transformation to calculate it.

Another approach to calculate finite temperature OTOC is to calculate $F(t-\iota\beta/4) = Tr(y^2VW(t)y^2VW(t))$ directly.

3.3 OTOC for twist fields in XXZ chain

From (81) we saw that the branch point twist fields have dimensions, $(h,\bar{h}) = (\frac{1}{8},\frac{1}{8})$. Hence the twist field operators are the same as Vertex operators with $\alpha = \pm 1/2$. In [26], it was shown that g scaling dimensions for the bulk correlation functions are given by

$$x^{(n)} = n^2\frac{\pi-\gamma}{2\pi} \tag{147}$$

where $n = 1,2,...$ represents the number of overturned spins from the anti-ferromagnetic ground state. Therefore we can find the corresponding operator in XXZ chain.

To calculate the OTOC, we consider the analytical continuation of (91), with $u_1 = v_2 = t$ and $u_2 = v_1 = 0$ where $x = 1$. For $\langle 0t0t\rangle$, we need

$$\begin{aligned} u_1 &= t + \iota\epsilon_1, \\ V_1 &= 0 + \iota\epsilon_2, \\ u_2 &= 0 + \iota\epsilon_3, \\ v_2 &= t + \iota\epsilon_4 \end{aligned} \tag{148}$$

with

$$\epsilon_3 > \epsilon_4 > \epsilon_2 > \epsilon_1 \tag{149}$$

Take $\epsilon_3 = 4\epsilon, \epsilon_4 = 3\epsilon, \epsilon_2 = 2\epsilon$ and $\epsilon_1 = \epsilon$. Then the cross ratio becomes

$$x' = 1 + \frac{4\epsilon^2}{t^2} - \frac{8\iota\epsilon^3}{t^3} + \mathcal{O}(\epsilon^4) \tag{150}$$

From (94), we see that the modular parameter goes from 0 to

$$\tau' = 0 - \frac{\pi\epsilon}{2t\left[\log\frac{2t}{\epsilon}\right]^2} + \frac{\iota\pi}{2\log\frac{2t}{\epsilon}} + \frac{\iota\pi(-2+\log\frac{16t^4}{\epsilon^4})\epsilon^2}{4t^2\left[\log\frac{2t}{\epsilon}\right]^3} + \mathcal{O}(\epsilon^3) \tag{151}$$

Then the crossratio term from partition function goes as

$$\frac{|u_1-u_2||v_1-v_2|}{|u_1-v_1||u_2-v_2||u_1-v_2||u_2-v_1|} = \frac{1}{\epsilon^2} + \frac{1}{t^2} + \frac{2\iota\epsilon}{t^3} + \mathcal{O}(\epsilon^2) \tag{152}$$

Note that the apparent singularities appearing will go away when we consider the full partition function. The CFT content, (92) can be calculated for certain values of η. For example, for $K = 1$ (or $\Delta = 0$), this gives a power law in t. Here we have used the modular property of Jacobi theta function, $\theta_i(-1/\tau) = \sqrt{-\iota\tau}\theta_i(\tau)$ because otherwise the series would have diverging terms and could not be calculated.

4 Summary and Discussion

We saw that the four-point function of the twist field is related to the partition function on $R_{2,2}$ which is isomorphic to a torus and calculated its partition function. We outlined the general process through which we can calculate the expectation value of energy momentum tensor for a general manifold $R_{n,N}$ and expectation value of energy-momentum tensor for $R_{2,2}$ for theories in which we know the partition function on the torus

We bosonized the spin-1/2 XXZ spin chain model to find that it is related to the Free bosonic theory compactified on a circle of radius R. Since the four-point function of twist fields on free bosonic CFT was known, we did the analytical continuation and found that the OTOC for $K = 1$ was a power law. This was expected because at zero inverse temperature there is no chaos in the system.

Future scope of the work would be to calculate OTOC corresponding to different values of Luttinger parameter (and hence Δ). We chose $K = 1$ in our work because at $K = 1$, the twist fields in the CFT correspond to spin-flip operators $S^{\pm}$ in the XXZ model. It would be interesting to calculate OTOC for other values of Luttinger parameter where twist fields correspond to interesting operators in CFT.

Another interesting observation would be to calculate the OTOC for finite temperature and match those with numerical calculations. At finite temperature, we can also hope to find the Lyapunov exponent to find chaos in the system, however, for spin-1/2 XXZ system described in this report, we suspect that there would be no non-trivial Lyapunov exponent (because the system has Poisson distribution of energy level spacing).

5 Appendix

5.1 Weiestrass function

The Weiestrass $\wp$-function is the sum of series

$$\wp(u) = \frac{1}{u^2} + \sum_{w \in L*} \left(\frac{1}{(u-w)^2} - \frac{1}{w^2} \right) \tag{153}$$

and

$$\wp'(u) = -2 \sum_{w \in L} \frac{1}{(u-w)^3} \tag{154}$$

with $\wp$ even and $\wp'$ odd, i.e., $\wp(-u) = \wp(u)$ and $\wp'(-u) = -\wp'(u)$. The Laurent expansion of the Weiestrass functions is given by

$$\begin{aligned} \wp(u) &= u^{-2} + 3G_4(L)u^2 + 5G_6(L)u^4 + ... \\ \wp'(u) &= -2u^{-3} + 6G_4(L)u + 20G_6(L)u^3 + ... \end{aligned} \tag{155}$$

5.2 Modular transformations of Dedekind-Eta function

Under T-transformations, $q \to e^{2\pi\iota}q$. Then the Dedekind eta function transforms as

$$\eta(\tau+1) = e^{\frac{\pi}{12}\iota}\eta(\tau) \tag{156}$$

Under S-transformation,

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-\iota\tau}\,\eta(\tau) \tag{157}$$

Hence, the partition function transforms as

$$\mathcal{Z}_{cir.}\left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}\right) = \mathcal{Z}_{cir.}(\tau, \bar{\tau}) \tag{158}$$

and

$$\mathcal{Z}_{cir.}(\tau, \bar{\tau}, 2/R) = \mathcal{Z}_{cir.}(\tau, \bar{\tau}, R) \tag{159}$$

5.3 Jacobi-Theta function

Jacobi theta functions are defined as

$$\begin{aligned} \theta_1(z|\tau) &= -\iota \sum_{r \in \mathbb{Z}+1/2} (-1)^{r-1/2} y^r q^{r^2/2} \\ \theta_2(z|\tau) &= \sum_{r \in \mathbb{Z}+1/2} y^r q^{r^2/2} \\ \theta_3(z|\tau) &= \sum_{n \in \mathbb{Z}} y^n q^{n^2/2} \\ \theta_1(z|\tau) &= -\iota \sum_{n \in \mathbb{Z}} (-1)^n y^n q^{n^2/2} \end{aligned} \tag{160}$$

where $q = e^{2\pi\iota\tau}$ and $y = e^{2\pi\iota z}$.

$$\begin{aligned} \theta_1(z+1|\tau) &= -\theta_1(z|\tau) \\ \theta_2(z+1|\tau) &= -\theta_2(z|\tau) \\ \theta_3(z+1|\tau) &= \theta_3(z|\tau) \\ \theta_4(z+1|\tau) &= \theta_4(z|\tau) \end{aligned} \tag{161}$$

Under modular transformations,

$$\begin{aligned} \theta_2(\tau+1) &= e^{\iota\pi/4}\theta_2(\tau) \\ \theta_3(\tau+1) &= \theta_4(\tau) \\ \theta_4(\tau+1) &= \theta_3(\tau) \end{aligned} \tag{162}$$

And

$$\begin{aligned} \theta_2\left(-\frac{1}{\tau}\right) &= \sqrt{-\iota\tau}\theta_4(\tau) \\ \theta_3\left(-\frac{1}{\tau}\right) &= \sqrt{-\iota\tau}\theta_3(\tau) \\ \theta_4\left(-\frac{1}{\tau}\right) &= \sqrt{-\iota\tau}\theta_2(\tau) \end{aligned} \tag{163}$$

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Acknowledgements

I would like to extend my sincerest gratitude to Professor P. Ramadevi for allowing me to carry out my Dual Degree Project under her efficient supervision. Her eminent guidance has helped increase my interest in gauge and gravity theories and it acts as a constant inspiration for me to pursue the answers to the problems of Physics. I would like to thank Professor Prithvi Narayan for his helpful discussions and collaboration. These discussions are the foundation of my understanding of CFT (special thanks for letting me use your 'Yellow book'). I would also like to thank Professor Urjit Yajnik, who gave me guidance on quantum field theory and initial inspiration to pursue research in theoretical physics. He has helped me by providing key guidance at important times. I also thank all other professors of Physics Department, Indian Institute of Technology, Bombay who have given their invaluable time in sharing their thoughts on various subjects. I would like to extend my sincere regard to my colleagues for their help and discussions on the topic.