Einstein-Cartan-Dirac Theory in Newmann-Penrose Formalism

B. Tech. 2 Project Report

by

Nehal Mittal

Roll Number: 14D260006

Guide:

Prof. T.P. Singh

Department of Astronomy and Astrophysics
Tata Institute of Fundamental Research

Co-guide:

Prof. Urjit Yajnik

Department of Physics
IIT Bombay

B. Tech and M. Tech: Dual Degree (Specialization: Nanoscience)

Department of Physics

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Report Approval Certificate

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Signature
(Supervisor : Prof. Urjit Yajnik)

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Signature
Name : Nehal Mittal
Roll Number : 14D260006
Date : December 1, 2019

Abstract

The Dirac equation on a *non-torsional* space-time (V_4) has been studied extensively in the NP formalism. In particular, a comprehensive treatment is given by S. Chandrasekhar in [6], which we take as our primary source. Building upon this work, we aim to (working always in the NP formalism) explicitly write the contorsion spin coefficients in terms of the Dirac spinor components, before generalising the Dirac equations by carrying them into a *torsional* space-time (U_4) – where it is known in this form as the *Hehl-Datta equation* – as permitted by the Einstein-Cartan-Sciama-Kibble (ECSK) framework which has nonvanishing torsion. Finally, we write down the full Einstein-Cartan-Dirac (ECD) equations in the NP formalism, and attempt a solution (on Minkowski background) in various specific cases.

Contents

Notation and conventions

The following conventions are in use for the remainder of this paper:

☐ The Lorentz Signature used is (+ - - -) throughout.

☐ V_4 is a non-torsional space-time, while a space-time endowed with torsion is specified by U_4.

☐ Greek indices, e.g. α, ζ, δ refer to world components, which transform according to *general coordinate transformations* and are raised or lowered using the metric $g_{\mu\nu}$.

☐ Latin indices within parenthesis e.g. (a) or (i) are tetrad indices, which transform according to *local Lorentz transformations* in the flat tangent space, and are raised or lowered using $\eta_{(i)(k)}$.

☐ Latin indices (without parenthesis) e.g. i, j, b, c indicate objects in Minkowski space, which transform according to *global Lorentz transformations*).

☐ In general $0, 1, 2, 3$ refer to world indices while $(0), (1), (2), (3)$ refer to tetrad indices.

☐ The total covariant derivative is denoted by ∇, while $\{\}$ denotes the Christoffel connection. Correspondingly, $\nabla^{\{\}}$ represents a covariant derivative with respect to the Christofell connections.

☐ Commas $(,)$ indicate partial derivatives while semicolons $(;)$ indicate the Riemannian covariant derivative. Thus, for tensors, ; and $\nabla^{\{\}}$ are same, while for spinors, $(;)$ involves both partial derivatives and the Riemannian part of the spin connection, γ, as defined in the following.

☐ The 4 component Dirac-spinor is written as

$$\psi = \begin{bmatrix} P^A \\ \bar{Q}_{B'} \end{bmatrix} \tag{1}$$

where P^A and $\bar{Q}_{B'}$ are two dimensional complex vectors in $\mathbb{C}^2$ space. We redefine the spinors as: $P^0 = F_1$, $P^1 = F_2$, $\bar{Q}^{1'} = G_1$ and $\bar{Q}^{0'} = -G_2$. This is in accordance with our primary source, [6], the notations, conventions and representations wherein are generally adhered to in this paper.

1 Introduction

Einstein's general theory of relativity (GR), published in 1915, has been described as the most beautiful of all the existing physical theories [1]. The background space-time on which classical GR is formulated is a Riemannian manifold (denoted V_4) which is torsion-less. In this case, the affine connection coincides uniquely with the Levi-Civita connection and geodesics coincide with the path of shortest distance. This is, however, not generally true for other, *torsional* manifolds, such as the manifold on which the Einstein-Cartan-Sciama-Kibble (ECSK) – or simple, Einstein-Cartan (EC) – theory is formulated. In such a theory, the geometrical structure of the manifold is modified such that the affine connection is no longer required to be symmetric, and no longer coincides uniquely with the Levi-Civita connection.

There are good reasons to believe that at very high ($\sim$GUT) energy scales – where the gravitational interaction becomes comparable in strength to the other quantum interactions[22], the current formulation of gravitation via General Relativity (GR) breaks down. Over the past century, there have been many attempts to reconcile gravity with the other fundamental interactions; ECSK theory is one such attempt[7, 3, 2, 8, 10, 9].

Torsion, as an antisymmetric part of the affine connection was introduced by Elie-Cartan (1922) [7]. Also termed the U_4 theories of gravitation, Einstein-Cartan theories work with an underlying manifold that is non-Riemannian (unlike classical GR which is formulated on V_4). The non-Riemannian part of the manifold is associated with the spin density of matter, which plays the role of a source analogous to the role of mass in Riemannian curvature. Here, mass and spin *both* play the dynamical role. While mass "adds up" on classical length scales due to its monopole character, spin, being of dipole character, usually averages out in the absence of external forces.

For this reason, matter, in the macro-physical regime, can be dynamically characterized entirely by the energy-momentum tensor. In the quantum-gravitational regime, heuristic arguments suggest that a spin density tensor plays an analogous role for spin, and related, as with mass and curvature, to some other geometrical property of space-time. It is this requirement that EC/ECSK theory satisfies (the reader is referred to my B.T.P. report 1 or [2] for a detailed treatment). When we minimally couple the Dirac field on U_4, we term this *Einstein-Cartan-Dirac (ECD) theory*. There are two independent geometric fields – the metric and torsion – and one matter field ψ in this theory. Varying the corresponding Lagrangian, we get three equations of motion, corresponding to the modified Einstein field equations, modified Dirac equation, and a torsional coupling. On U_4, the Dirac equation on U_4 becomes non-linear; we call this the *Hehl-Datta (HD) equation* after the seminal work in [3].

The usual method in approaching solutions to problems in GR is to use a *local coordinate basis* $\hat{e}^\mu$ such that $\hat{e}^\mu = \partial_\mu$. This coordinate basis field is covariant under general coordinate transformations. However, it has been found useful to employ non-coordinate basis techniques in problems involving spinors. Moreover, choosing the tetrad basis vectors as *null vectors* is extremely useful in some situations. This formalism, where a given theory is expressed in the basis of null tetrads, is the celebrated Newman-Penrose (NP) formalism. In this formalism, we replace tensors by their null tetrad components and represent these components with certain distinctive symbols(a detailed study of this was done as a part of B.T.P. 1 and can be found in B.T.P. report 1). Most of the important and physically relevant geometrical objects and identities (eg. the Riemann curvature tensor, Weyl tensor, Bianchi identities, Ricci identities etc.) on U_4 have been formulated in the NP formalism (such as in [14]).

It can be shown that there is a natural connection between spin dyads (Part of B.T.P. 1) and null tetrads [6, 13]. Physical systems involving spinor fields can be fully expressed in the NP formalism (for example, the Dirac equation on V_4 has been studied extensively, ref. Chapter 12 in [6]). In addition, many systems in gravitational physics are also studied in the NP formalism [6]. It appears that the NP formalism is the shared vocabulary between the physics of quantum mechanical systems (with spinor fields) and classical gravitational systems (having a metric and/or torsion).

In the present paper, we aim to formulate the full ECD equations in the NP formalism. We know that the contorsion tensor is completely expressible in terms of the Dirac state [2]. We wish to then find expressions for the contorsion spin coefficients – which are the standard NP variables that account for spin – explicitly in terms of the Dirac state. Using this, we can write the complete set of HD equations in the NP formalism. In a sense, this work is to be read as a sequel to the work of S. Chandrasekhar in (Chapter 12 of) [6], where Dirac equation in V_4 has been given a full treatment in the NP formalism. Some recent works attempt to do that but have not provided explicit corrections to the standard NP variables due to torsion. Further, there are notational and sign inconsistencies in many such examples of existing literature on in the field, and we aim to provide a comprehensive and self-contained treatment.

Finally, we attempt at solutions to the HD equations in a Minkowski space with torsion. This, apart from being the simplest case to consider, is also motivated by certain physical intuitions which can be considered as supporting, but non-essential, corollaries to this work. A recent essay, [4, 5, 11], suggests the incorporation of new length scale in quantum gravity, thereby providing a symmetry between large and small masses; a conjecture has been proposed therein to establish a duality between these two limits. This conjecture is predicated on the necessary existence of solutions to the Hehl-Datta equations on Minkowski space, representing the balance between the Riemannian and torsional effects which reduce to small and large masses in the respective limits. However, notwithstanding the duality conjecture and the new length scale proposed, our results hold for standard theory as well. All equations are expressed in terms of a generic length scale l, which is what takes on different values in the case of the standard theory and in the case of modified theories with new length scales.

2 Einstein-Cartan theory and its coupling to the Dirac field

2.1 Einstein-Cartan theory

In the Einstein-Cartan theory, the Riemannian manifold of ordinary GR (V_4) is promoted to the corresponding non-Riemannian manifold U_4. As discussed, this latter manifold admits, in addition to the structure of ordinary GR, a non-vanishing torsion. Torsion is a (rank 3) tensorial object defined as the antisymmetric part of the affine connection:

$$Q_{\alpha\beta}{}^{\mu} = \Gamma_{[\alpha\beta]}{}^{\mu} = \frac{1}{2}(\Gamma_{\alpha\beta}{}^{\mu} - \Gamma_{\beta\alpha}{}^{\mu}) \tag{2}$$

Similarly, the *contorsion* tensor $K_{\alpha\beta}{}^{\mu}$ is given by $K_{\alpha\beta}{}^{\mu} = -Q_{\alpha\beta}{}^{\mu} - Q^{\mu}{}_{\alpha\beta} + Q_{\beta}{}^{\mu}{}_{\alpha}$. This allows us to write – in terms of the usual Christoffel symbols – the following relation:

$$\Gamma_{\alpha\beta}{}^{\mu} = \left\{ {\mu \atop \alpha\beta} \right\} - K_{\alpha\beta}{}^{\mu} \tag{3}$$

The covariant derivative is then defined by

$$\nabla_{\alpha} B^{\mu} = \partial_{\mu} B^{\mu} + \Gamma_{\alpha\beta}{}^{\mu} B^{\beta} \tag{4}$$

When a matter field ψ is minimally coupled with gravity and torsion, its action is given as follows[2]:

$$S = \int d^4x \sqrt{-g} \Big[\mathcal{L}_m(\psi, \nabla\psi, g) - \frac{1}{2k} R(g, \partial g, Q) \Big] \tag{5}$$

Here $k = 8\pi G/c^4$, $\mathcal{L}_m$ is the matter Lagrangian density, and the second term represents the Lagrangian density for the gravitational field. There are three fields in this Lagrangian: ψ, $g_{\mu\nu}$, and $K_{\alpha\beta\mu}$, representing the matter field, the metric, and the contorsion, respectively. Varying the action with respect to these, one arrives at the following three field equations:

$$\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta\psi} = 0 \tag{6}$$

$$\frac{\delta(\sqrt{-g}R)}{\delta g^{\mu\nu}} = 2k\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}} \tag{7}$$

$$\frac{\delta(\sqrt{-g}R)}{\delta K_{\alpha\beta\mu}} = 2k\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta K_{\alpha\beta\mu}} \tag{8}$$

Here, (6) leads us to the matter field equations on a curved space-time with torsion. The right hand side of (7) is associated with $\sqrt{-g}kT_{\mu\nu}$ via the definition of $T_{\mu\nu}$, the metric energy-momentum tensor. Similarly, the right hand side of (8) is associated with $2\sqrt{-g}kS^{\mu\beta\alpha}$ where $S^{\mu\beta\alpha}$ is the spin density tensor. Together, these two yield the Einstein-Cartan field equations:

$$G^{\mu\nu} = k\Sigma^{\mu\nu} \tag{9}$$

$$T^{\mu\beta\alpha} = kS^{\mu\beta\alpha} \tag{10}$$

In (9) the $G^{\mu\nu}$ on the left hand side is the asymmetric Einstein tensor built from the asymmetric connection, while $\Sigma^{\mu\nu}$ is the asymmetric canonical (total) energy momentum tensor, constructed out of the symmetric (metric) energy-momentum tensor and the spin density tensor. In (10), the so-called 'modified' torsion $T^{\mu\beta\alpha}$ is the traceless part of the torsion tensor, and is algebraically related to $S^{\mu\beta\alpha}$ on the right. These three are related toeach other by:

$$\Sigma^{\mu\nu} = T^{\mu\nu} + (\nabla_\alpha + 2Q_\alpha)[S^{\mu\nu\alpha} - S^{\nu\alpha\mu} + S^{\alpha\mu\nu}] \tag{11}$$

In the limit torsion $\to 0$, we recover classical GR – (10) vanishes, and (9) reduces to the Einstein field equations which couple the (symmetric) Einstein tensor to the (symmetric) metric energy-momentum tensor.

2.2 EC coupling to the Dirac field

The theory generated from the minimal coupling of the Dirac field on U_4 is what we term *Einstein-Cartan-Dirac (ECD) theory*. In this theory, the matter field is the spinorial Dirac field ψ, for which the Lagrangian density is given by (note the noncommuting covariant derivatives):

$$\mathcal{L}_m = \frac{i\hbar c}{2}(\overline{\psi}\gamma^\mu\nabla_\mu\psi - \nabla_\mu\overline{\psi}\gamma^\mu\psi) - mc^2\overline{\psi}\psi \tag{12}$$

In ECD theory, the addition of spin degrees of freedom necessitates a more careful treatment of anholonomic objects. As we define the affine connection, Γ, to facilitate parallel transport of geometrical objects with world (Greek) indices, so do we define the spin connection γ for anholonomic objects (with Latin indices). The affine connection can be decomposed into a Riemannian ($\{\}$) and a torsional part (made up of the contorsion tensor, K) and similarly, the spin connection γ can also be decomposed into a Riemannian (γ^o) and torsional part (once again, formed of the contorsion tensor). These components are related via the following equation (following the notation in [14]):

$$\gamma_\mu{}^{(i)(k)} = \gamma_\mu^{o(i)(k)} - K_\mu{}^{(k)(i)} \tag{13}$$

where $\gamma_\mu^{o(i)(k)}$ is Riemannian part and $K_\mu{}^{(k)(i)}$ is the torsional part. Using these, we define the covariant derivative for spinors, on V_4 and U_4:

$$\psi_{;\mu} = \partial_\mu\psi + \frac{1}{4}\gamma^o_{\mu(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\psi \qquad \text{(on } V_4\text{)} \tag{14}$$

$$\nabla_\mu\psi = \partial_\mu\psi + \frac{1}{4}\gamma^0_{\mu(c)(b)}\gamma^{[(b)}\gamma^{(c)]}\psi - \frac{1}{4}K_{\mu(c)(b)}\gamma^{[(b)}\gamma^{(c)]}\psi \qquad \text{(on } U_4\text{)} \tag{15}$$

Substituting this into (12) we obtain the explicit form of Lagrangian density; varying with respect to $\bar{\psi}$ as in (6) yields the Dirac equation on V_4 and U_4:

$$i\gamma^{\mu}\psi_{;\mu} - \frac{mc}{\hbar}\psi = 0 \qquad \text{(on } V_4\text{)} \tag{16}$$

$$i\gamma^{\mu}\psi_{;\mu} + \frac{i}{4}K_{(a)(b)(c)}\gamma^{[(a)}\gamma^{(b)}\gamma^{(c)]}\psi - \frac{mc}{\hbar}\psi = 0 \qquad \text{(on } U_4\text{)} \tag{17}$$

Next, we use (7) and Lagrangian density defined in (12) to obtain the gravitational field equations on V_4 and U_4:

$$G_{\mu\nu}(\{\}) = \frac{8\pi G}{c^4}T_{\mu\nu} \qquad \text{(on } V_4\text{)} \tag{18}$$

$$G_{\mu\nu}(\{\}) = \frac{8\pi G}{c^4}T_{\mu\nu} - \frac{1}{2}\left(\frac{8\pi G}{c^4}\right)^2 g_{\mu\nu}S^{\alpha\beta\lambda}S_{\alpha\beta\lambda} \qquad \text{(on } U_4\text{)} \tag{19}$$

Here, $T_{\mu\nu}$ is the dynamical EM tensor which is symmetric and defined as:

$$T_{\mu\nu} = \Sigma_{(\mu\nu)}(\{\}) = \frac{i\hbar c}{4}\left[\bar{\psi}\gamma_{\mu}\psi_{;\nu} + \bar{\psi}\gamma_{\nu}\psi_{;\mu} - \bar{\psi}_{;\mu}\gamma_{\nu}\psi - \bar{\psi}_{;\nu}\gamma_{\mu}\psi\right] \tag{20}$$

Equations (16) and (18) together form the system of equations of Einstein-Dirac theory.

We now move to the full Einstein-Cartan-Dirac theory . Using the Lagrangian density defined in (12), we can define the spin density tensor:

$$S^{\mu\nu\alpha} = \frac{-i\hbar c}{4}\bar{\psi}\gamma^{[\mu}\gamma^{\nu}\gamma^{\alpha]}\psi \tag{21}$$

Using (21) and (8) , (17) can be simplified to give the Hehl-Datta equation [2][3]. This, along with (19) and the relation between the modified torsion tensor and spin density tensor, define the field equations of the Einstein-Cartan-Dirac theory:

$$G_{\mu\nu}(\{\}) = \frac{8\pi G}{c^4}T_{\mu\nu} - \frac{1}{2}\left(\frac{8\pi G}{c^4}\right)^2 g_{\mu\nu}S^{\alpha\beta\lambda}S_{\alpha\beta\lambda} \tag{22}$$

$$T_{\mu\nu\alpha} = -K_{\mu\nu\alpha} = \frac{8\pi G}{c^4}S_{\mu\nu\alpha} \tag{23}$$

$$i\gamma^{\mu}\psi_{;\mu} = +\frac{3}{8}L_{Pl}^2\overline{\psi}\gamma^5\gamma_{(a)}\psi\gamma^5\gamma^{(a)}\psi + \frac{mc}{\hbar}\psi \tag{24}$$

$$\tag{25}$$

where L_{Pl} is the Planck length.

3 Introducing a unified length scale L_{CS} in quantum gravity

Both Dirac theory and general relativity claim to hold for all values of m and it is only through experiments that we find that Dirac equation holds if $m \ll m_{Pl}$ while Einstein equations hold if $m \gg m_{Pl}$ From the theoretical viewpoint, it is unsatisfactory that the two theories should have to depend on the experiment to establish their domain of validity. Therefore a more general length scale was proposed by recent works of T. P. Singh [4][5] where motivation has been provided for unifying the Compton wavelength ($\lambda_c = \frac{\lambda}{\hbar c}$) and Schwarzschild radius ($R_s = \frac{2GM}{c^2}$) of a point particle with mass m into one single length scale, the Compton-Schwarzschild length (L_{CS}). Such a treatment compells us to introduce torsion, and identify the Dirac field with the complex torsion field. An action principle has been proposed with this new length scale which permits the Dirac equation and the Einstein field equations as mutually dual limiting cases. The modified action proposed is as follows:

$$\frac{L_{Pl}^2}{\hbar}S = \int d^4x\sqrt{-g}\left[R - \frac{1}{2}L_{CS}\bar{\psi}\psi + L_{CS}^2\bar{\psi}i\gamma^\mu\partial_\mu\psi\right] \tag{26}$$

Using this new length scale, L_{CS}, we can rewrite the Einstein-Cartan-Dirac equations as[5]:

$$G_{\mu\nu}(\{\}) = \frac{8\pi L_{CS}^2}{\hbar c}T_{\mu\nu} + \left(\frac{8\pi L_{CS}^2}{\hbar c}\right)^2 \tau_{\mu\nu} \tag{27}$$

$$T_{\mu\nu\gamma} = \frac{8\pi L_{CS}^2}{\hbar c}S_{\mu\nu\gamma} \tag{28}$$

$$i\gamma^a\psi_{;a} = +\frac{3}{8}L_{CS}^2\bar{\psi}\gamma^5\gamma_a\psi\gamma^5\gamma^a\psi + \frac{1}{2L_{CS}}\psi = 0 \tag{29}$$

A note on length scales: We use the l to denote a length scale in the theory. For standard ECD theory, the typical scales that can be considered are the Planck length, with $l = L_{Pl} = \sqrt{\frac{G\hbar}{c^3}}$, half the Compton wavelength, with $l_1 = \frac{\lambda_C}{2} = \frac{\hbar}{2mc}$, or the Schwarzschild radius, with $l_2 = R_s = \frac{2GM}{c^2}$. For the modified ECD theory, we take $l = L_{CS}$, in terms of the new unified length scale.

This system of equations is the same as the standard ECD equations, except that in Eqns.(27) and (28) l_{Pl} is replaced by L_{CS} in Eqns. (29) the L_{Pl} in the nonlinear term, and the λ_c in the mass term are both replaced by L_{CS}. These equations hold for all values of the mass m, and it is then only natural that the coupling constant should be L_{CS}, instead of L_{Pl} and λ_c, for why should the latter two appear in the ECD equations for a large mass?

3.1 Duality between Curvature and Torsion

We notice from the above that curvature vanishes in the small mass limit, whereas in the large mass limit it is torsion that vanishes, because GR holds in the large mass limit. This motivates us to ask if this kind of curvature - torsion duality could be generic. Indeed, we have remarked earlier that since L_{CS} is the only coupling constant in the theory, it will label a large mass solution of the field equations, and also label its dual small mass solution. However, we expect the large mass solution to be gravity dominated, and the small mass solution to be torsion dominated. This is possible only if for a given L_{CS} there are two solutions, one that is curvature dominated, and another that is torsion dominated. We call this the curvature - torsion duality, and construct it as follows.The total curvature R on a space-time manifold can be written as a sum of contribution Q because of torsion and an additional the Reimannean curvature R^0 made from the symmetric Levi-Civita connection:

$$R^\rho{}_{\theta\mu\nu} = R^{0\rho}{}_{\theta\mu\nu} + Q^\rho{}_{\theta\mu\nu} := 0 \tag{30}$$

where $Q^\rho{}_{\theta\mu\nu} = \overset{\{\}}{\nabla}_\mu K^\rho{}_{\theta\nu} - \overset{\{\}}{\nabla}_\nu K^\rho{}_{\theta\mu} + K^\sigma{}_{\theta\nu}K^\rho{}_{\sigma\mu} - K^\sigma{}_{\theta\mu}K^\rho{}_{\sigma\nu}$ and $R^{0\rho}{}_{\theta\mu\nu}$ is the curvature of Levi-Civita connection, K is the contortion tensor and the covariant derivative is with respect to the Levi-Civita connection.

Suppose we have a curvature dominated large mass solution $S1$ with a given L_{CS} and the set of curvature parameters $[R(1), R^0(1), Q(1)]$. We define the dual torsion dominated solution S2 having the same L_{CS} and the set of curvature parameters $[R(2), R^0(2), Q(2)]$ by the following map:

$$R(1) - Q(1) = Q(2) - R(2) \tag{31}$$

which means that the excess of curvature over torsion for S1 equals the excess of torsion over curvature for S2. This duality implies that $R^0(2) = -R^0(1)$. In the large mass limit, $Q(1)$ is zero and we have the pure curvature solution $R(1) = R^0(1)$ which is the V_4 theory. In the small mass limit, $R(2)$ is zero, and we have the solution $Q(2) = -R^0(2)$. Since $R(2)$ is zero, this is teleparallel gravity, and the duality map implies that $R(1) = Q(2)$. This duality provides an intriguing connection between GR, ECSK theory, and teleparallel gravity. The first and third theories are respectively the large mass and small mass limit of the ECSK theory and are connected by a duality. We have provided a symmetry between curvature and torsion, and provided a physical basis for Poincare gauge gravity.

This is qualitatively depicted in Fig. (1) where $R - Q$ is plotted against $z = \ln{(m/m_{Pl})}$. The dual masses M and m have the same value of L_{CS}, and the curvature dominated solution $S1$ in the first quadrant is mapped to the torsion dominated solution $S2$ in the third quadrant. As the mass is reduced, a solution 'rolls down' from the first quadrant to the origin m_{Pl} and transits to the solution set in the third quadrant. There is also a 'mirror universe' whose significance remains to be investigated: For a given L_{CS} the curvature dominated large mass solution is also realized for the dual small mass. This provides the mirror solution which rolls down from the second to the fourth quadrant, and where small masses are curvature dominated, while large masses are torsion dominated. At the transition point $m = m_{Pl}$ we have $R - Q = 0$ so that $R^0 = 0$: this is a Minkowski at space-time where the total curvature is sourced only by torsion. One might ask if it is justified to couple the quantum mechanical Dirac field with classical curvature. The answer is that the Dirac field is quantum only when curvature and torsion are neglected.

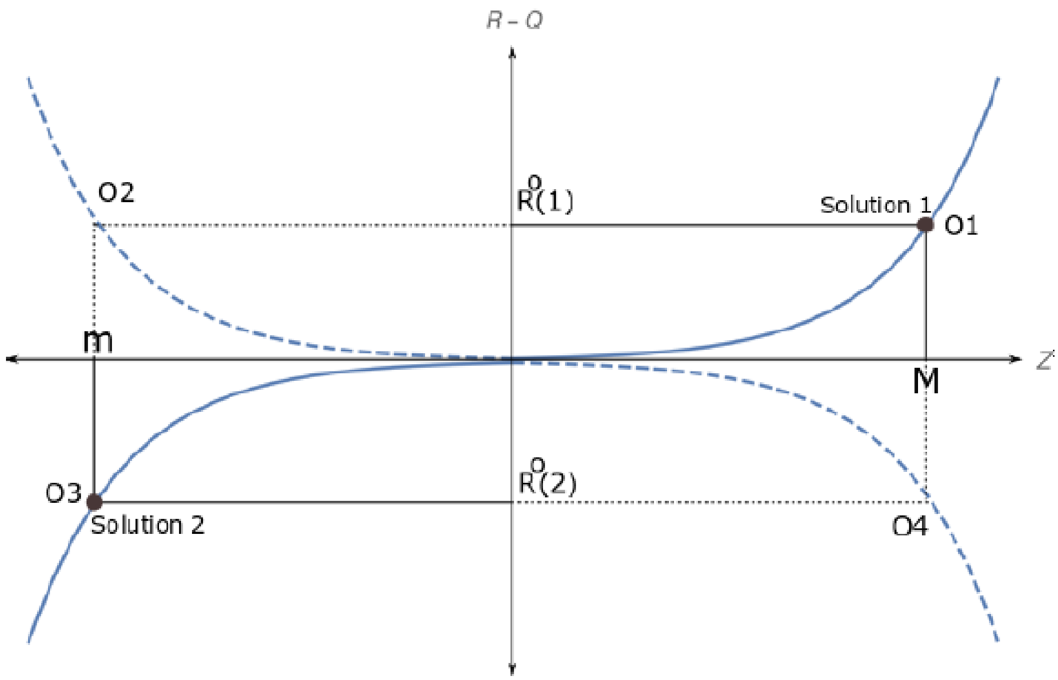

Figure 1: The curvature - torsion duality

Furthermore, as we saw, the coupling constant G arises only when the mass becomes large, as if to suggest that gravity emerges only when sufficient mass has accumulated, and then gravity is strictly classical. Only torsion has to be second quantized, which makes sense, because torsion is explicitly expressed in terms of the Dirac state.

4 The Newman-Penrose formalism and ECD in NP

4.1 Tetrads

It is common in the literature[6] to use tetrads (or *vierbeins*) to define spinors on a curved space-time (in V_4 as well as U_4)[1]. In this formalism, the transformation properties of spinors are defined in a flat (Minkowski) space, locally tangent to U_4. We know that at each point in space-time, we can define a coordinate basis vector field $\hat{e}^\mu = \partial_\mu$ which is covariant under general coordinate transformations. The basis vectors associated with spinors, however, are covariant under *local* Lorentz transformations. Hence, we define, at each point of our manifold, a set of four orthonormal basis vectors (forming the tetrad field) given by $\hat{e}^i(x)$. These comprise four vectors (one for each μ) at each point, and the tetrad field is governed by the relation $\hat{e}^i(x) = e^i_\mu(x)\hat{e}^\mu$ where the transformation matrix e^i_μ is such that:

$$e^{(i)}_\mu e^{(k)}_\nu \eta_{(i)(k)} = g_{\mu\nu} \tag{32}$$

The transformation matrix $e^{(i)}_\mu$ allows us to convert the components of any world tensor (a tensor which transforms according to general coordinate transformations) to the corresponding components in a local Minkowskian space (the latter of these being covariant under local Lorentz transformations). Directional derivatives of a smooth scalar field ϕ in the direction of e_a is defined as the tangent contravariant vectors

$$e_{(a)}(\phi) := e_{(a)}{}^\mu \frac{\partial\phi}{\partial x^\mu} \tag{33}$$

For the components of a vector field,

$$A_{(a),(b)} = A_{(a);(b)}\gamma_{(c)(a)(b)}A^{(c)} \tag{34}$$

where $\gamma_{(c)(b)(a)}$ are the Ricci rotation coefficients which are anti-symmetric in first pair of indices and are defined as the rotation of $e_{(a)}$ when dragged along $e(b)$ with respect to $e_{(c)}$

$$\gamma_{(c)(a)(b)} := e_{(c)}{}^\rho e_{(a)\rho;\mu} e_{(b)}{}^\mu = \eta_{(c)(d)}\gamma^{(d)}{}_{(a)(b)} \tag{35}$$

In tetrad formalism, Ricci identities, Bianchi identities and the Lie bracket (which itself is a tangent vector and hence can be written in the same basis as $e_{(a)}$) are respectively

$$\begin{aligned} R_{(a)(b)(c)(d)} = \gamma_{(a)(b)(c),(d)} + \gamma_{(a)(b)(d),(c)} + \gamma_{(b)(a)(f)}[\gamma_{(c)}{}^{(f)}{}_{(d)} - \gamma_{(d)}{}^{(f)}{}_{(c)}] \\ +\gamma_{(f)(a)(c)}\gamma_{(b)}{}^{(f)}{}_{(d)} - \gamma_{(f)(a)(d)}\gamma_{(b)}{}^{(f)}{}_{(c)} \end{aligned} \tag{36}$$

$$\begin{aligned} R_{(a)(b)[(c)(d)|(f)]} = \frac{1}{6}\sum_{[(c)(d)(f)]} R_{(a)(b)(c)(d),(f)} - \eta^{(n)(m)}[\gamma_{(n)(a)(f)}R_{(m)(b)(c)(d)} + \gamma_{(n)(b)(f)}R_{(a)(m)(c)(d)} \\ +\gamma_{(n)(c)(f)}R_{(a)(b)(m)(d)} + \gamma_{(n)(d)(f)}R_{(a)(b)(c)(m)}] \end{aligned} \tag{37}$$

$$[e_{(a)}, e_{(b)}] = C^{(c)}{}_{(a)(b)}e_{(c)} \tag{38}$$

where $C^{(c)}{}_{(a)(b)}$ is called the structure constant and can be written as

$$C^{(c)}{}_{(a)(b)} = \gamma^{(c)}{}_{(b)(a)} - \gamma^{(c)}{}_{(a)(b)} \tag{39}$$

There are 36 Ricci identities (34), 20 Bianchi identities (35) and 24 commutation relations (36).

[1]While this is often the case, there are other formalisms that can be used[23]

4.2 Introduction to the NP formalism

The Newman-Penrose (NP) formalism was formulated by Newman and Penrose in their work [15]. It is a special case of tetrad formalism; where we choose our tetrad as a set of four orthonormal null vectors:

$$e^{\mu}_{(0)} = l^{\mu}, \quad e^{\mu}_{(1)} = n^{\mu}, \quad e^{\mu}_{(2)} = m^{\mu}, \quad e^{\mu}_{(3)} = \bar{m}^{\mu} \tag{40}$$

where l^{μ}, n^{μ} are real and $m^{\mu}, \bar{m}^{\mu}$ are complex. The orthonormality condition on null tetrads imply

$$\begin{aligned} &l.m = l.\bar{m} = n.m = n.\bar{m} = 0, \\ &l.l = n.n = m.m = \bar{m}.\bar{m} = 0, \\ &l.n = 1 \qquad and \qquad m.\bar{m} = -1 \end{aligned} \tag{41}$$

The null tetrad indices are raised and lowered using the flat space-time metric

$$\eta_{(i)(j)} = \eta^{(i)(j)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{42}$$

and the tetrad vectors satisfy the equation $g_{\mu\nu} = e^{(i)}_{\mu} e^{(j)}_{\nu} \eta_{(i)(j)}$. In this formalism, we replace tensors by their tetrad components and represent these components with a collection of distinctive symbols which are now standard in literature which can be found in [6] and [14] for V_4 and U_4 respectively.

4.2.1 Petrov Classification

Rotations in tetrad frame are of three major types:

Type I: $l \to l, m \to m + al, \bar{m} = \bar{m} + a^* l$ and $n \to n + a^* m + a\bar{m} + aa^* l$

Type II: $n \to n$, $m \to m + bn$, $\bar{m} \to \bar{m} + b^* n$ and $l \to l + b^* m + b\bar{m} + bb^* n$

Type III: $l \to A^{-1} l, n \to An, m \to e^{\iota\theta} m$ and $\bar{m} \to e^{-\iota\theta}\bar{m}$

where a and b are two complex functions and A and θ are two real functions.
From [6] and [14], we find that the form of Ψ_0 remains same in V_4 and U_4 theory. Under rotation of type II with b as a parameter, Ψ_0 transforms to

$$\Psi_0^{(1)} = \Psi_0 + 4b\Psi_1 + 6b^2\Psi_3 + b^4\Psi_4 \tag{43}$$

This can be made 0 by choosing b to be a root of this equation. The corresponding new directions are called principle null-directions and based on the number of distinct roots, we make the following classifications:

∗ If all the roots of this equation are distinct, then by a rotation of Type II, Ψ_0 can be made 0. By rotation of Type I, we can make Ψ_4 vanish. Such solutions are called ***Petrov Type I***.

∗ If two roots are coincident, then by a rotation of Type II,Ψ_0 and Ψ_1 can be made 0. By rotation of Type I, we can make Ψ_4 vanish. Such solutions are called ***Petrov Type II***.

∗ If equation (2.45) has two distinct double roots, then by a rotation of Type II, followed by rotation of Type I, we can make Ψ_0, Ψ_1, Ψ_3 and Ψ_4 vanish. Such solutions are called ***Petrov Type D***.

∗ If three roots are coincident, then by a rotation of Type II, Ψ_0, Ψ_1 and Ψ_2 can be made 0. By rotation of Type I, we can make Ψ_4 vanish. Such solutions are called ***Petrov Type III***.

∗ If all four roots are coincident, then by a rotation of Type II, we can make Ψ_0, Ψ_1, Ψ_2 and Ψ_3 vanish. Such solutions are called ***Petrov Type N***.

A much more rigorous analysis for this in V_4 can be found in [6]

4.3 Spinor analysis

We define four null tetrads (and their corresponding co-vectors) on Minkowski space (raised and lowered using $\eta_{\mu\nu}$):

$$l^a = \frac{1}{\sqrt{2}}(1,0,0,1),\ m^a = \frac{1}{\sqrt{2}}(0,1,-i,0),\ \bar{m}^a = \frac{1}{\sqrt{2}}(0,1,i,0),\ n^a = \frac{1}{\sqrt{2}}(1,0,0,-1) \tag{44}$$

We also define the following Van der Waarden symbols:

$$\sigma^a = \sqrt{2}\begin{bmatrix} l^a & m^a \\ \bar{m}^a & n^a \end{bmatrix} \qquad \tilde{\sigma}^a = \sqrt{2}\begin{bmatrix} n^a & -m^a \\ -\bar{m}^a & l^a \end{bmatrix} \tag{45}$$

For the Dirac gamma matrices, we use the complex version of the Weyl (chiral) representation:

$$\gamma^a = \begin{bmatrix} 0 & (\tilde{\sigma}^a)^* \\ (\sigma^a)^* & 0 \end{bmatrix} \quad \text{where} \quad \gamma^0 = \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix},\ \gamma^i = \begin{bmatrix} 0 & (-\sigma^i)^* \\ (\sigma^i)^* & 0 \end{bmatrix} \tag{46}$$

where $a = (0,1,2,3)$.

The complex Weyl representation is used so that the Dirac bispinor and gamma matrices defined in equation 1 and 46 remains consistent with equations (97) and (98) of section (103) in [6] (comparing with our standard reference, [6], we recover equation (99) in complex form).

In order to represent spinorial objects (objects comprising spinors and gamma matrices) on a curved space-time, we use the following prescription on the tetrad formalism[13], viz. – Let $\mathcal{M}$ be a curved manifold with all conditions necessary for the existence of spin structure, and let U be a chart on $\mathcal{M}$ with coordinate functions (x^α). Then, for representing spinorial objects, we (i) choose an orthonormal tetrad field $e^\mu_{(a)}(x^\alpha)$ on U, (ii) define the Van der Waarden symbols $\sigma^{(a)}$ and $\tilde{\sigma}^{(a)}$ in this tetrad basis exactly as defined on Minkowski space in (45) and choose a γ representation (46); (iii) then, the σ's in a local coordinate frame are then obtained via:

$$\sigma^\mu(x^\alpha) = e^\mu_{(a)}(x^\alpha)\sigma^{(a)} = \sqrt{2}\begin{bmatrix} l^\mu & m^\mu \\ \bar{m}^\mu & n^\mu \end{bmatrix} \qquad \tilde{\sigma}^\mu = e^\mu_{(a)}(x^\alpha)\tilde{\sigma}^{(a)} = \sqrt{2}\begin{bmatrix} n^\mu & -m^\mu \\ -\bar{m}^\mu & l^\mu \end{bmatrix} \tag{47}$$

with the γ matrices obeying a similar transformation.

Thus, objects with world indices (containing world-indexed γ matrices or spinors) are now functions of chosen orthonormal tetrads. These are defined *a priori* in a local tetrad basis (with components identical to those defined on a flat Minkowski space-time) and *then* carried into a curved space via the tetrads. This is unlike other geometrical world objects which are first defined naturally at a point in a manifold and subsequently carried to a local tangent space via tetrads.

We now aim to carry the Dirac equation (in NP) on V_4 into the U_4 space, building upon Section 102(d) of [6]. In order to calculate the covariant derivative of a spinor in U_4, we require the spinor affine connection coefficients. They are defined via the requirement that ϵ_{AB} and σ's are covariantly constant. The analysis in [6] – until Eq. 91 in the book – still stands; however, the covariant derivatives are promoted to those acting on U_4. They are defined as follows::

$$\nabla_\mu P^A = \partial_\mu P^A + \Gamma^A_{\mu B} P^B \tag{48}$$

$$\nabla_\mu \bar{Q}^{A'} = \partial_\mu \bar{Q}^{A'} + \bar{\Gamma}^{A'}_{\mu B'} \bar{Q}^{B'} \tag{49}$$

The Γ terms here are added to the partial derivative when working with objects in U_4. Their values can completely be determined in terms of the spin coefficients, and we can readily evaluate its tetrad components. Using *Friedman's lemma* (see pg. 542 of [6] for a full proof), we can express the various spin coefficients $\Gamma_{(a)(b)(c)(d')}$ in terms of covariant derivatives of the basis null vectors l, n, m and $\bar{m}$. The covariant derivative here is exactly as defined in equation Eq. 3.3 (and explicitly written in Eq. 3.5) of [14].

Using this covariant derivative, it is readily seen how Eq. 95 and Eq. 96 in [6] get modified; viz. $\Gamma_{0000'} = \kappa^o + \kappa_1$ and $\Gamma_{1101'} = \mu^o + \mu_1$ (Naughts in the superscript are used to indicate the original spin coefficients defined on V_4). The 12 independent spin coefficients are calculated in terms of covariant derivatives of null vectors and defined in the following table[2] (50):

$\Gamma_{(a)(b)(c)(d')} =$ (50)

(c)(d') \ (a)(b)	00	01 or 10	11
00'	$\kappa^o + \kappa_1$	$\epsilon^o + \epsilon_1$	$\pi^o + \pi_1$
10'	$\rho^o + \rho_1$	$\alpha^o + \alpha_1$	$\lambda^o + \lambda_1$
01'	$\sigma^o + \sigma_1$	$\beta^o + \beta_1$	$\mu^o + \mu_1$
11'	$\tau^o + \tau_1$	$\gamma^o + \gamma_1$	$\nu^o + \nu_1$

4.4 Contorsion spin coefficients in terms of Dirac spinor components

The spin density tensor of matter ($S^{\mu\nu\lambda}$) can be written as a world tensor in U_4 made up of the Dirac spinor, its adjoint, and gamma matrices:

$$S^{\mu\nu\alpha} = \frac{-i\hbar c}{4}\bar{\psi}\gamma^{[\mu}\gamma^{\nu}\gamma^{\alpha]}\psi \tag{51}$$

The ECD field equations suggest that $T^{\mu\nu\alpha} = kS^{\mu\nu\alpha}$ where $T^{\mu\nu\alpha}$ is the modified torsion tensor defined in Eq. 2.3 of [2]. It can be shown that, for Dirac field, $T^{\mu\nu\alpha} = -K^{\mu\nu\alpha} = kS^{\mu\nu\alpha}$ as in Eq. 5.6 of [3]. Here, k is a gravitational coupling constant containing the length scale l, i.e., $\frac{8\pi l^2}{\hbar c}$. For the standard theory, $l = L_{Pl}$ and for modified theory, $l = L_{CS}$. Substituting (51) in the field equations, we obtain following:

$$K^{\mu\nu\alpha} = -kS^{\mu\nu\alpha} = 2i\pi l^2\bar{\psi}\gamma^{[\mu}\gamma^{\nu}\gamma^{\alpha]}\psi \tag{52}$$

where the γ^μ's are those defined in (46), generalised with world indices using orthonormal tetrads. We subsequently rewrite the completely anti-symmetric tensor, $K^{\mu\nu\alpha}$ (of which only four independent components are excited by the Dirac field) in the NP formalism; i.e., in the null tetrad basis, as follows:

$$K_{(i)(j)(k)} = e_{(i)\mu}e_{(j)\nu}e_{(k)\alpha}K^{\mu\nu\alpha} \tag{53}$$

where $e_{(i)\mu} = (l_\mu, n_\mu, m_\mu, \bar{m}_\mu)$ for $i = 1, 2, 3, 4$ To calculate the contorsion spin coefficients, we need to evaluate the contorsion tensor with world indices as defined in (171). Consider the product $\gamma^\alpha\gamma^\beta\gamma^\mu$, which is defined as:

$$\gamma^\alpha\gamma^\beta\gamma^\mu = \begin{pmatrix} 0 & (\tilde{\sigma}^\alpha)^*(\sigma^\beta)^*(\tilde{\sigma}^\mu)^* \\ (\sigma^\alpha)^*(\tilde{\sigma}^\beta)^*(\sigma^\mu)^* & 0 \end{pmatrix} \tag{54}$$

The explicit form of this matrix is fairly expansive, and a full treatment is given in Appendix A. Eventually, we substitute in for the Dirac bispinor (as defined in [6]), and obtain the expressions for the contorsion spin coefficients in terms of the spinor components. We have, for example, for ρ –

$$\rho = -K_{(1)(3)(4)} = -2\sqrt{2}i\pi l^2[F_2\bar{F}_2 - G_1\bar{G}_1] \tag{55}$$

All the contorsion spin coefficients can be found in a similar fashion. After evaluating those, the eight non-zero spin coefficients excited by the Dirac spinor given in (1) – of which four are independent – are as follows:

$$\tau_1 = -2\beta_1 = K_{123} = 2\sqrt{2}i\pi l^2(F_2\bar{F}_1 + G_2\bar{G}_1) \tag{56}$$

$$\pi_1 = -2\alpha_1 = K_{124} = 2\sqrt{2}i\pi l^2(-F_1\bar{F}_2 - G_1\bar{G}_2) \tag{57}$$

$$\mu_1 = -2\gamma_1 = -K_{234} = 2\sqrt{2}i\pi l^2(F_1\bar{F}_1 - G_2\bar{G}_2) \tag{58}$$

$$\rho_1 = -2\epsilon_1 = -K_{134} = 2\sqrt{2}i\pi l^2(G_1\bar{G}_1 - F_2\bar{F}_2) \tag{59}$$

(60)

[2] In the generic case, all 12 have contorsion spin coefficients

From the above relations, we have:

$$\mu_1 = -\mu_1^* \tag{61}$$

$$\rho_1 = -\rho_1^* \tag{62}$$

$$\pi_1 = +\tau_1^* \tag{63}$$

The table in (50) is modified as follows:

$\Gamma_{(a)(b)(c)(d')} =$ (64)

(c)(d') \ (a)(b)	00	01 or 10	11
00'	κ_0	$\epsilon_0 - \rho_1/2$	$\pi_0 + \pi_1$
10'	$\rho_0 + \rho_1$	$\alpha_0 - \pi_1/2$	λ_0
01'	σ_0	$\beta_0 - \tau_1/2$	$\mu_0 + \mu_1$
11'	$\tau_0 + \tau_1$	$\gamma_0 - \mu_1/2$	ν_0

4.5 The Dirac equation with torsion in the NP formalism

The Dirac equation on U_4 (also known as the Hehl-Datta equation) is:

$$i\gamma^\mu \nabla_\mu \psi = \frac{mc}{\hbar}\psi = \frac{\psi}{2l} \tag{65}$$

where ∇ here denotes covariant derivative on U_4. $l = \frac{\lambda_c}{2}$ for standard theory and $l = L_{cs}$ for modified theory. It can be written in the following matrix form:

$$i\begin{pmatrix} 0 & (\tilde{\sigma}^\mu)^* \\ (\sigma^\mu)^* & 0 \end{pmatrix} \nabla_\mu \begin{pmatrix} P^A \\ \bar{Q}_{B'} \end{pmatrix} = \frac{1}{2\sqrt{2}l}\begin{pmatrix} P^A \\ \bar{Q}_{B'} \end{pmatrix} \tag{66}$$

This can be written as a pair of matrix equations:

$$\begin{pmatrix} \sigma^\mu_{00'} & \sigma^\mu_{10'} \\ \sigma^\mu_{01'} & \sigma^\mu_{11'} \end{pmatrix} \nabla_\mu \begin{pmatrix} P^0 \\ P^1 \end{pmatrix} + \frac{i}{2\sqrt{2}l}\begin{pmatrix} -\bar{Q}^{1'} \\ \bar{Q}^{0'} \end{pmatrix} = 0 \tag{67}$$

$$\begin{pmatrix} \sigma^\mu_{11'} & -\sigma^\mu_{10'} \\ -\sigma^\mu_{01'} & \sigma^\mu_{00'} \end{pmatrix} \nabla_\mu \begin{pmatrix} -\bar{Q}^{1'} \\ \bar{Q}^{0'} \end{pmatrix} + \frac{i}{2\sqrt{2}l}\begin{pmatrix} P^0 \\ P^1 \end{pmatrix} = 0 \tag{68}$$

Working out explicitly, the first equation is:

$$\begin{aligned} \frac{i}{2\sqrt{2}l}\bar{Q}^{1'} &= \sigma^\mu_{00'}\nabla_\mu P^0 + \sigma^\mu_{10'}\nabla_\mu P^1 = (\partial_{00'}P^0 + {\Gamma^0}_{i00'}P^i) + (\partial_{10'}P^1 + {\Gamma^1}_{i10'}P^i) \\ &= (D + {\Gamma^0}_{000'}P^0 + {\Gamma^0}_{100'}P^1) + (\delta^* + {\Gamma^1}_{010'}P^0 + {\Gamma^1}_{110'}P^1) \\ \Rightarrow \frac{i}{2\sqrt{2}l}G_1 &= (D + \epsilon_0 - \rho_0)F_1 + (\delta^* + \pi_0 - \alpha_0)F_2 + \frac{3}{2}(\pi_1 F_2 - \rho_1 F_1) \end{aligned} \tag{69}$$

where we have used the gamma matrices as defined in (46), computed the covariant derivatives using (48), (49) and the spin connections in terms of contorsion spin coefficients as given in (64). Using this procedure (full treatment given in Appendix B), the four Dirac equations are obtained as:

$$(D + \epsilon_0 - \rho_0)F_1 + (\delta^* + \pi_0 - \alpha_0)F_2 + \frac{3}{2}(\pi_1 F_2 - \rho_1 F_1) = ib(l)G_1 \tag{70}$$

$$(\Delta + \mu_0 - \gamma_0)F_2 + (\delta + \beta_0 - \tau_0)F_1 + \frac{3}{2}(\mu_1 F_2 - \tau_1 F_1) = ib(l)G_2 \tag{71}$$

$$(D + \epsilon_0^* - \rho_0^*)G_2 - (\delta + \pi_0^* - \alpha_0^*)G_1 - \frac{3}{2}(\tau_1 G_1 - \rho_1 G_2) = ib(l)F_2 \tag{72}$$

$$(\Delta + \mu_0^* - \gamma_0^*)G_1 - (\delta^* + \beta_0^* - \tau_0^*)G_2 - \frac{3}{2}(\mu_1 G_1 - \pi_1 G_2) = ib(l)F_1 \tag{73}$$

Substituting in the spinorial form of the contorsion spin coefficients in (56) - (60), we obtain:

$$(D+\epsilon_0-\rho_0)F_1+(\delta^*+\pi_0-\alpha_0)F_2+ia(l)[(-F_1\bar{F}_2-G_1\bar{G}_2)F_2+(F_2\bar{F}_2-G_1\bar{G}_1)F_1]=ib(l)G_1 \quad (74)$$
$$(\Delta+\mu_0-\gamma_0)F_2+(\delta+\beta_0-\tau_0)F_1+ia(l)[(F_1\bar{F}_1-G_2\bar{G}_2)F_2-(F_2\bar{F}_1+G_2\bar{G}_1)F_1]=ib(l)G_2 \quad (75)$$
$$(D+\epsilon_0^*-\rho_0^*)G_2-(\delta+\pi_0^*-\alpha_0^*)G_1-ia(l)[(F_2\bar{F}_2-G_1\bar{G}_1)G_2+(F_2\bar{F}_1+G_2\bar{G}_1)G_1]=ib(l)F_2 \quad (76)$$
$$(\Delta+\mu_0^*-\gamma_0^*)G_1-(\delta^*+\beta_0^*-\tau_0^*)G_2-ia(l)[(F_1\bar{F}_1-G_2\bar{G}_2)G_1-(-F_1\bar{F}_2-G_1\bar{G}_2)G_2]=ib(l)F_1 \quad (77)$$

where $a(l)=3\sqrt{2}\pi l^2$, $b(l)=\frac{1}{2\sqrt{2}l}$.

These equations can be condensed into the following form:

$$(D+\epsilon_0-\rho_0)F_1+(\delta^*+\pi_0-\alpha_0)F_2=i[b(l)+a(l)\xi]G_1 \quad (78)$$
$$(\Delta+\mu_0-\gamma_0)F_2+(\delta+\beta_0-\tau_0)F_1=i[b(l)+a(l)\xi]G_2 \quad (79)$$
$$(D+\epsilon_0^*-\rho_0^*)G_2-(\delta+\pi_0^*-\alpha_0^*)G_1=i[b(l)+a(l)\xi^*]F_2 \quad (80)$$
$$(\Delta+\mu_0^*-\gamma_0^*)G_1-(\delta^*+\beta_0^*-\tau_0^*)G_2=i[b(l)+a(l)\xi^*]F_1 \quad (81)$$

where $\xi=F_1\bar{G}_1+F_2\bar{G}_2$ and $\xi^*=\bar{F}_1G_1+\bar{F}_2G_2$.

4.6 The dynamical EM tensor $T_{\mu\nu}$ and spin density in the NP formalism

The equation of interest here is (19); given as: $G_{\mu\nu}(\{\})=\frac{8\pi l^2}{\hbar c}T_{\mu\nu}-\frac{1}{2}\left(\frac{8\pi l^2}{\hbar c}\right)^2 g_{\mu\nu}S^{\alpha\beta\lambda}S_{\alpha\beta\lambda}$. The dynamical EM tensor on U_4 is given by equation 20. In the above equation, the second term in the RHS is given as $\frac{4\pi l^2}{\hbar c}g_{\mu\nu}S^{\alpha\beta\gamma}S_{\alpha\beta\gamma}$ which can now be written as

$$\frac{4\pi l^2}{\hbar c}g_{\mu\nu}S^{\alpha\beta\gamma}S_{\alpha\beta\gamma}=6\pi\hbar c l^2 g_{\mu\nu}(F_1\bar{G}_1+F_2\bar{G}_2)(\bar{F}_1G_1+\bar{F}_2G_2)=6\pi\hbar c l^2 g_{\mu\nu}\xi\xi^* \quad (82)$$

i.e., proportional to the ζ parameter introduced.

4.7 Einstein Energy-Momentum Relations in NP formalism

We set up two coordinate systems on 4-dimensional Minkowski space -orthonormal coordinate system (x_1,x_2,x_3,x_4) and null coordinate system (r,s,u,v). The partial derivatives w.r.t these coordinates in terms of Newman-Penrose variables are

$$\partial_1=\frac{D+\Delta}{\sqrt{2}},\partial_2=\frac{\delta+\delta^*}{\sqrt{2}},\partial_3=\frac{i(\delta-\delta^*)}{\sqrt{2}},\partial_4=\frac{D-\Delta}{\sqrt{2}}, \qquad and \quad (83)$$

$$\partial_r=D,\partial_s=\Delta,\partial_u=\delta,\partial_v=\delta^*,\partial_r=D,\partial_s=\Delta,\partial_u=\delta,\partial_v=\delta^* \quad (84)$$

The EM tensor T_{ij} which appears on the RHS of equation (4.10) is the Riemann part of symmetrized dynamical EM tensor. On Minkowski space, it is given as

$$T_{ij}=\Sigma_{(ij)}(\{\})=\frac{i\hbar c}{4}\left(\bar{\psi}\gamma_i\partial_j\psi+\bar{\psi}\gamma_j\partial_i\psi-\partial_i\bar{\psi}\gamma_j\psi-\partial_j\bar{\psi}\gamma_i\psi\right) \quad (85)$$

We find the value of this tensor in both, orthonormal as well as the null system of coordinates. In both the systems, we express the tensor in Newman-Penrose variables (D, Δ, etc.)

5 Solutions to HD equations in Minkowski space

5.1 Motivation

In the previous section, we formulated the ECD equations in the NP formalism. In this section, we aim to solve them. The simplest space-time with torsion is the Minkowski ($\eta_{\mu\nu}$) space-time with a manifold that has non-zero torsion. In this space-time, the Dirac equation on U_4 looks very similar to the linear Dirac equation with modified mass (the torsion-related term which modifies it is bilinear in the Dirac states). In this spirit, we will consider modifications (due to torsion) to well-studied solutions to the linear Dirac equation (eg. plane wave solutions).

In addition, there are good (physical) reasons to work within Minkowski space-time, to find solution(s) of the HD equations incorporating torsion. In a recent work [4, 5, 11], a duality between large and small masses (correspondingly, between Riemannian curvature and torsion) has been proposed, explicitly constructed in the "curvature-torsion duality conjecture" therein. For this conjecture to hold true, a solution to Dirac equation on Minkowski space with torsion must exist – along with certain other conditions. One such additional condition is the vanishing $(T-S)_{\mu\nu}$ tensor, as defined in Appendix C.

While we proceed in the following section to find solutions to the HD equations on Minkowski space for their own sake, the reader may find, in [11], useful extensions to this work. To this end, in the Appendices (ref. Appendix C) we have also computed the $(T-S)_{\mu\nu}$ tensor in certain cases, for completeness.

5.2 The Hehl-Datta equations on Minkowski space with torsion

The HD equations on Minkowski space with torsion (in the NP formalism) are as follows:

$$DF_1 + \delta^* F_2 = i[b(l) + a(l)\xi]G_1 \tag{86}$$
$$\Delta F_2 + \delta F_1 = i[b(l) + a(l)\xi]G_2 \tag{87}$$
$$DG_2 - \delta G_1 = i[b(l) + a(l)\xi^*]F_2 \tag{88}$$
$$\Delta G_1 - \delta^* G_2 = i[b(l) + a(l)\xi^*]F_1 \tag{89}$$

In a Cartesian coordinate system $(ct, x, y, z)^3$ we have:

$$(\partial_0 + \partial_3)F_1 + (\partial_1 + i\partial_2)F_2 = i\sqrt{2}[b(l) + a(l)\xi]G_1 \tag{90}$$
$$(\partial_0 - \partial_3)F_2 + (\partial_1 - i\partial_2)F_1 = i\sqrt{2}[b(l) + a(l)\xi]G_2 \tag{91}$$
$$(\partial_0 + \partial_3)G_2 - (\partial_1 - i\partial_2)G_1 = i\sqrt{2}[b(l) + a(l)\xi^*]F_2 \tag{92}$$
$$(\partial_0 - \partial_3)G_1 - (\partial_1 + i\partial_2)G_2 = i\sqrt{2}[b(l) + a(l)\xi^*]F_1 \tag{93}$$

In cylindrical polar coordinates (ct, r, ϕ, z), we have:

$$r\partial_t F_1 + e^{i\phi} r\partial_r F_2 + ie^{i\phi}\partial_\phi F_2 + r\partial_z F_1 = ir\sqrt{2}[b(l) + a(l)\xi]G_1 \tag{94}$$
$$r\partial_t F_2 + e^{-i\phi} r\partial_r F_1 - ie^{-i\phi}\partial_\phi F_1 - r\partial_z F_2 = ir\sqrt{2}[b(l) + a(l)\xi]G_2 \tag{95}$$
$$r\partial_t G_2 - e^{-i\phi} r\partial_r G_1 + ie^{-i\phi}\partial_\phi G_1 + cr\partial_z G_2 = ir\sqrt{2}[b(l) + a(l)\xi^*]F_2 \tag{96}$$
$$r\partial_t G_1 - e^{i\phi} r\partial_r G_2 - ie^{i\phi}\partial_\phi G_2 - r\partial_z G_1 = ir\sqrt{2}[b(l) + a(l)\xi^*]F_1 \tag{97}$$

Likewise, in spherical polar coordinates (ct, r, θ, ϕ):

[3] Setting $c = 1$ by convention

$$\partial_t F_1 + \cos\theta\partial_r F_1 - \frac{\sin\theta}{r}\partial_\theta F_1 + \frac{ie^{i\phi}}{r\sin\theta}\partial_\phi F_2 + e^{i\phi}\sin\theta\partial_r F_2 + \frac{e^{i\phi}\cos\theta}{r}\partial_\theta F_2 = i\sqrt{2}[b(l) + a(l)\xi]G_1 \tag{98}$$

$$\partial_t F_2 - \cos\theta\partial_r F_2 - \frac{\sin\theta}{r}\partial_\theta F_2 + \frac{ie^{-i\phi}}{r\sin\theta}\partial_\phi F_1 + e^{-i\phi}\sin\theta\partial_r F_1 - \frac{e^{-i\phi}\cos\theta}{r}\partial_\theta F_1 = i\sqrt{2}[b(l) + a(l)\xi]G_2 \tag{99}$$

$$\partial_t G_2 + \cos\theta\partial_r G_2 - \frac{\sin\theta}{r}\partial_\theta G_2 - \frac{ie^{-i\phi}}{r\sin\theta}\partial_\phi G_1 - e^{-i\phi}\sin\theta\partial_r G_1 + \frac{e^{-i\phi}\cos\theta}{r}\partial_\theta G_1 = i\sqrt{2}[b(l) + a(l)\xi^*]F_2 \tag{100}$$

$$\partial_t G_1 - \cos\theta\partial_r G_1 - \frac{\sin\theta}{r}\partial_\theta G_1 - \frac{ie^{i\phi}}{r\sin\theta}\partial_\phi G_2 - e^{i\phi}\sin\theta\partial_r G_2 - \frac{e^{i\phi}\cos\theta}{r}\partial_\theta G_2 = i\sqrt{2}[b(l) + a(l)\xi^*]F_1 \tag{101}$$

6 Solutions to Hehl-Datta equations on Minkowski space-time

6.1 Attempting a non-static solution by reduction of the problem to 1+1 dimension

Assuming the Dirac state to be a function of only t and z, and further assuming an ansatz of the form $F_1 = G_2$ and $F_2 = G_1$, the four equations in Cartesian coordinates [90 - 93] as well as four equations in cylindrical coordinates [94 - 97] reduce to following two independent equations[4]

$$\partial_t\psi_1 + \partial_z\psi_2 - i\sqrt{2}b\psi_1 + \frac{ia}{\sqrt{2}}(|\psi_2|^2 - |\psi_1|^2)\psi_1 = 0 \tag{102}$$

$$\partial_t\psi_2 + \partial_z\psi_1 + i\sqrt{2}b\psi_2 + \frac{ia}{\sqrt{2}}(|\psi_1|^2 - |\psi_2|^2)\psi_2 = 0 \tag{103}$$

where, $\psi_1 = F_1 + F_2$ and $\psi_2 = F_1 - F_2$. We put $\sqrt{2}b = -m$ and $a = 2\sqrt{2}\lambda$ and obtain following:

$$\partial_t\psi_1 + \partial_z\psi_2 + im\psi_1 + 2i\lambda(|\psi_2|^2 - |\psi_1|^2)\psi_1 = 0 \tag{104}$$

$$\partial_t\psi_2 + \partial_z\psi_1 - im\psi_2 + 2i\lambda(|\psi_1|^2 - |\psi_2|^2)\psi_2 = 0 \tag{105}$$

This is identical to Eqn. 1 in [21]. This work by Alvarez et.al is the study of the convergence and stability of the difference scheme for the non-linear Dirac equation in 1+1 dimension. Following this work, the solutions to above set of equations are found using the following solitary wave as ansatz:

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}A(z)\\ iB(z)\end{pmatrix} e^{-i\Lambda t} \tag{106}$$

where A and B are real functions of z. Substituting this into the above equations, we obtain:

$$B' - (\sqrt{2}b + \Lambda)A - \frac{a}{\sqrt{2}}(A^2 - B^2)A = 0 \tag{107}$$

$$A' - (\sqrt{2}b - \Lambda)B - \frac{a}{\sqrt{2}}(A^2 - B^2)B = 0 \tag{108}$$

Solving these differential equations gives the following solutions for $A(z)$ and $B(z)$.

[4] We note that $\xi = 2Re(F_1\bar{F}_2)$, thus $\xi = \xi^*$. Furthermore, a and b are henceforth shorthand for $a(l)$ and $b(l)$.

$$A(z) = \frac{-i2^{3/4}(\sqrt{2}b - \Lambda)}{\sqrt{a}} \frac{\sqrt{(\sqrt{2}b + \Lambda)}\cosh(z\sqrt{2b^2 - \Lambda^2})}{[\Lambda\cosh(2z\sqrt{2b^2 - \Lambda^2}) - \sqrt{2}b]} \tag{109}$$

$$B(z) = \frac{-i2^{3/4}(\sqrt{2}b + \Lambda)}{\sqrt{a}} \frac{\sqrt{(\sqrt{2}b - \Lambda)}\sinh(z\sqrt{2b^2 - \Lambda^2})}{[\Lambda\cosh(2z\sqrt{2b^2 - \Lambda^2}) - \sqrt{2}b]} \tag{110}$$

As mentioned, this is the generalization of the equations for $A(z)$ and $B(z)$ in [21] (see Section III). Putting $\lambda = 0.5$ (equivalently $a = \sqrt{2}$) and $m = 1$ (equivalently $m_0 = -1$) in (109), (110), reduces to the equations on page 9 of [21]. This solution is also found in [16] with $a(l_1) = a(L_{Pl})$ and $b(l_2) = b(\lambda_c)$.

$$F_1 = G_2 = \frac{\sqrt{(2b^2 - \Lambda^2)}}{2}\left[\frac{-i2^{3/4}}{\sqrt{a}} \frac{\sqrt{(\sqrt{2}b - \Lambda)}\cosh(z\sqrt{2b^2 - \Lambda^2})}{[\Lambda\cosh(2z\sqrt{2b^2 - \Lambda^2}) - \sqrt{2}b]} + \frac{2^{3/4}}{\sqrt{a}} \frac{\sqrt{(\sqrt{2}b + \Lambda)}\sinh(z\sqrt{2b^2 - \Lambda^2})}{[\Lambda\cosh(2z\sqrt{2b^2 - \Lambda^2}) - \sqrt{2}b]}\right] e^{-i\Lambda t} \tag{111}$$

$$F_2 = G_1 = \frac{\sqrt{(2b^2 - \Lambda^2)}}{2}\left[\frac{-i2^{3/4}}{\sqrt{a}} \frac{\sqrt{(\sqrt{2}b - \Lambda)}\cosh(z\sqrt{2b^2 - \Lambda^2})}{[\Lambda\cosh(2z\sqrt{2b^2 - \Lambda^2}) - \sqrt{2}b]} - \frac{2^{3/4}}{\sqrt{a}} \frac{\sqrt{(\sqrt{2}b + \Lambda)}\sinh(z\sqrt{2b^2 - \Lambda^2})}{[\Lambda\cosh(2z\sqrt{2b^2 - \Lambda^2}) - \sqrt{2}b]}\right] e^{-i\Lambda t} \tag{112}$$

$$\xi = \frac{-2\sqrt{2}(2b^2 - \Lambda^2)(\sqrt{2}b - \Lambda\cosh(2z\sqrt{2b^2 - \Lambda^2})}{a[\Lambda\cosh(2z\sqrt{2b^2 - \Lambda^2}) - \sqrt{2}b]^2} \tag{113}$$

The probability density is given by the zeroth component of the four-vector fermion current $j^\mu = \bar{\psi}\gamma^\mu\psi$. Hence, it is $j^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\psi$. For our solution given above, it is given by

$$j^0 = \psi^\dagger\psi = 2\Big(|F_1|^2 + |F_2|^2\Big) \tag{114}$$

$$= \Big(|A|^2 + |B|^2\Big) \tag{115}$$

We will now define new dimensionless variables:

$$p = \sqrt{2}bz \qquad [p] = 0 \tag{116}$$

$$w = -\frac{\Lambda}{\sqrt{2}b} \qquad [w] = 0 \tag{117}$$

$$\tilde{A}(z) = \frac{\sqrt{a}}{2\sqrt{b}}A(z) \qquad [\tilde{A}(z)] = 0 \tag{118}$$

$$\tilde{B}(z) = \frac{\sqrt{a}}{2\sqrt{b}}B(z) \qquad [\tilde{B}(z)] = 0 \tag{119}$$

$$\tilde{j}^0 = \frac{a}{4b}j^0 \qquad [\tilde{j}^0] = 0 \tag{120}$$

With these definitions. A(p) and B(p) take the following form:

$$A(z) = \frac{2i(1+w)}{\sqrt{a}} \frac{\sqrt{b(1-w)}\cosh(p\sqrt{1-w^2})}{(w\cosh(2p\sqrt{1-w^2}) + 1)} \tag{121}$$

$$B(z) = \frac{2i(1-w)}{\sqrt{a}} \frac{\sqrt{b(1+w)}\sinh(p\sqrt{1-w^2})}{(w\cosh(2p\sqrt{1-w^2}) + 1)} \tag{122}$$

There are total six different cases for the values of w which gives different solutions. In each case, we will analyze the limit of zero-torsion also [That is the case of linear Dirac equation]. Vanishing of torsion is characterized by the limit $a(l_2) = 3\sqrt{2}\pi L_{Pl}^2 \longrightarrow 0$. So in a torsion-less case, differential equations become: (we have incorporated dimensionless constants in these equations)

$$B' = (1-w)A \tag{123}$$

$$A' = (1+w)B \tag{124}$$

- **Case 1**: $w \in (-\infty, -1)$

$$\tilde{A}(z) = i(1+w)\frac{\sqrt{(|w|+1)}\cos(p\sqrt{w^2-1})}{(1-|w|\cos(2p\sqrt{w^2-1}))} \tag{125}$$

$$\tilde{B}(z) = i(w-1)\frac{\sqrt{(|w|-1)}\sin(p\sqrt{w^2-1})}{(1-|w|\cos(2p\sqrt{w^2-1})} \tag{126}$$

$$\frac{a}{4b}j^0 = \tilde{j}^0 = \left[\frac{(w+1)^2(|w|+1)\cos^2(p\sqrt{w^2-1}) + (w-1)^2(|w|-1)\sin^2(p\sqrt{w^2-1})}{(1-|w|\cos(2p\sqrt{w^2-1})^2}\right] \tag{127}$$

Comments on case 1: This solution has infinite singularities placed periodically over non-zero values of 'p'. It is clearly unphysical. A specimen of this case with (w=-2) has been plotted in the left column of fig (4)

Contrasting case 1 with linear (non-torsion) Dirac equation: For $w \in (-\infty, -1)$, the linear Dirac equation gives plane waves solutions. The probability density fluctuates in a sinusoidal fashion. Plane wave solutions are physically meaningful. Addition of torsion, however, as we saw, makes this solution unphysical. For (w=-2), we have plotted this solution in fig (5).

- **Case 2**: $w = \pm 1$ (trivial case)

$$\tilde{A}(z) = 0 \qquad \tilde{B}(z) = 0 \qquad \tilde{j}^0 = 0 \tag{128}$$

- **Case 3**: $w \in (-1, 0)$

$$\tilde{A}(z) = i(1+w)\frac{\sqrt{(1+|w|)}\cosh(p\sqrt{1-w^2})}{(1-|w|\cosh(2p\sqrt{1-w^2})} \tag{129}$$

$$\tilde{B}(z) = i(1-w)\frac{\sqrt{(1-|w|)}\sinh(p\sqrt{1-w^2})}{(1-|w|\cosh(2p\sqrt{1-w^2})} \tag{130}$$

$$\frac{a}{4b}j^0 = \tilde{j}^0 = \left[\frac{(w+1)^2(|w|+1)\cosh^2(p\sqrt{1-w^2}) + (1-w)^2(1-|w|)\sinh^2(p\sqrt{1-w^2})}{(1-|w|\cosh(2p\sqrt{1-w^2})^2}\right] \tag{131}$$

Comments on case 3: This solution has two singularities placed symmetrically opposite w.r.t origin on two finite non-zero values of 'p'. It dies down to zero at infinity. However, owing to the presence of singularities, it is an unphysical solution. A specimen of this case with (w=-0.5) has been plotted in the left column of fig (2)

Contrasting case 3 with linear (non-torsion) Dirac equation: For $w \in (-1, 0)$, the linear Dirac equation has unphysical solutions. The solutions exponentially increase to infinity as 'p' go to $\pm\infty$. For, w=-0.5, we have plotted this solution in fig (5). So for case 3, we conclude that both linear (non-torsional) and non-linear (with torsion) Dirac equation give unphysical solutions.

- **Case 4**: $w = 0$

$$\tilde{A}(z) = i\cosh(p), \qquad \tilde{B}(z) = i\sinh(p), \qquad \tilde{j}^0 = \left[\cosh^2(p) + \sinh^2(p)\right] \tag{132}$$

Comments on case 4: This solution exponentially blows up at $p = \pm\infty$. Hence, it is clearly unphysical solution. A specimen of this case with (w=0) has been plotted in the right column of fig (2)

Contrasting case 4 with linear (non-torsion) Dirac equation: For $w = 0$, the linear Dirac equation is unphysical. The solutions exponentially increase to infinity as 'p' go to $+\infty$. For, w=0, we have plotted this solution in fig (5). So for case 4, we conclude that both linear (non-torsional) and non-linear (with torsion) Dirac equation give unphysical solutions.

- **Case 5**: $w \in (0, 1)$

$$\tilde{A}(z) = i(1+w)\frac{\sqrt{(1-w)}\cosh(p\sqrt{1-w^2})}{(1+w\cosh(2p\sqrt{1-w^2}))} \tag{133}$$

$$\tilde{B}(z) = i(1-w)\frac{\sqrt{(1+w)}\sinh(p\sqrt{1-w^2})}{(1+w\cosh(2p\sqrt{1-w^2}))} \tag{134}$$

$$\frac{a}{4b}j^0 = \tilde{j}^0 = \left[\frac{(1+w)^2(1-w)\cosh^2(p\sqrt{1-w^2}) + (1-w)^2(1+w)\sinh^2(p\sqrt{1-w^2})}{(1+w\cosh(2p\sqrt{1-w^2}))^2}\right] \tag{135}$$

Comments on case 5: In this case, we have no singularities anywhere. All the functions (including probability density) are asymptotically vanishing. So, It represents a physically plausible solution. Depending on the exact nature of solution, we classify this case into two sub-cases:
1) Case 5(a): $w \in (0, \frac{1}{3})$
2) Case 5(b): $w \in [\frac{1}{3}, 1)$

Case 5(a) has a local minima at the origin and has two global maximas on the two symmetrically opposite sides of origin at non-zero p's. Specimen of this case is shown in figure (3) with a blue plot. Case 5(b) has global maxima at origin and it monotonically decreases to zero at infinity. Two specimens of this case are shown in figure (3) with a orange and green plot. case 5(b) is like a 'blob' solution. We will comment more about this in 'discussions'.

Contrasting case 5 with linear (non-torsion) Dirac equation: For $w \in (0, 1)$, linear Dirac equation gives unphysical solution. The solutions exponentially increase to infinity as 'p' go to $\pm\infty$. For, w=0.5, we have plotted this solution in fig (5). Adding torsion in the picture, as we saw in this case, make the solutions physically meaningful.

- **Case 6**: $w \in (1, \infty)$

$$\tilde{A}(z) = -(1+w)\frac{\sqrt{(w-1)}\cos(p\sqrt{w^2-1})}{(1+w\cos(2p\sqrt{w^2-1})} \tag{136}$$

$$\tilde{B}(z) = -(1-w)\frac{\sqrt{(w+1)}\sin(p\sqrt{w^2-1})}{(1+w\cos(2p\sqrt{w^2-1})} \tag{137}$$

$$\frac{a}{4b}j^0 = \tilde{j}^0 = \left[\frac{(1+w)^2(w-1)\cos^2(p\sqrt{w^2-1}) + (1-w)^2(w+1) + \sin^2(p\sqrt{w^2-1})}{(1+w\cos(2p\sqrt{w^2-1}))^2}\right] \tag{138}$$

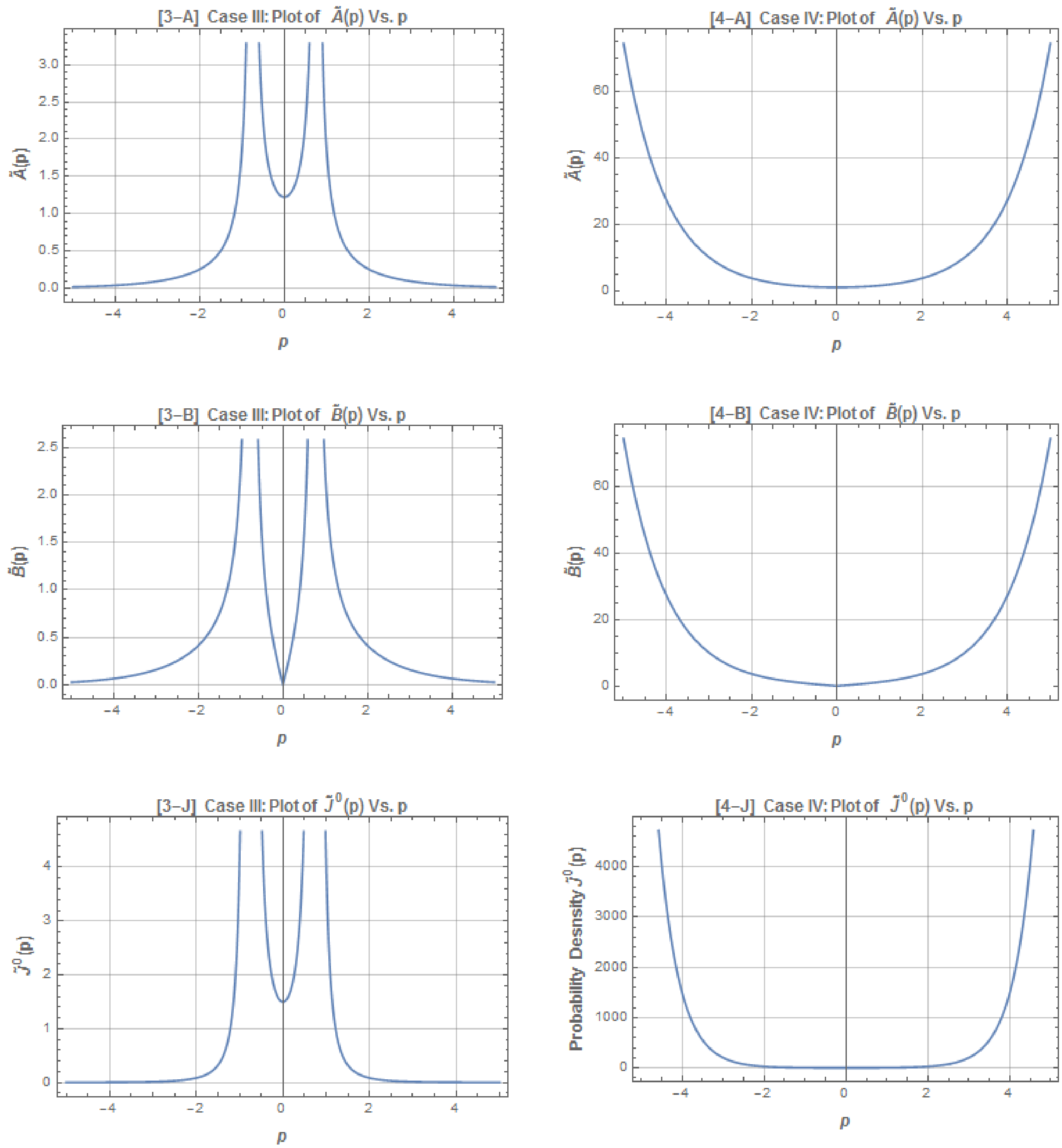

Figure 2: **Plots for case [III] and case [IV]**. The left column shows plots for case 3 with w=-0.5. The right column shows plots for case 4 with w=0. Both the cases have unphysical solutions.

Comments on case 3: This solution has infinite singularities placed periodically over non-zero values of 'p'. It is clearly unphysical. A specimen of this case with (w=2) has been plotted in the left column of fig (4)

Contrasting case 3 with linear (non-torsion) Dirac equation: For $w \in (1, \infty)$, the linear Dirac equation gives plane waves solutions. The probability density fluctuates in a sinusoidal fashion. Plane wave solutions are physically meaningful. Addition of torsion, however, as we saw, makes this solution unphysical.

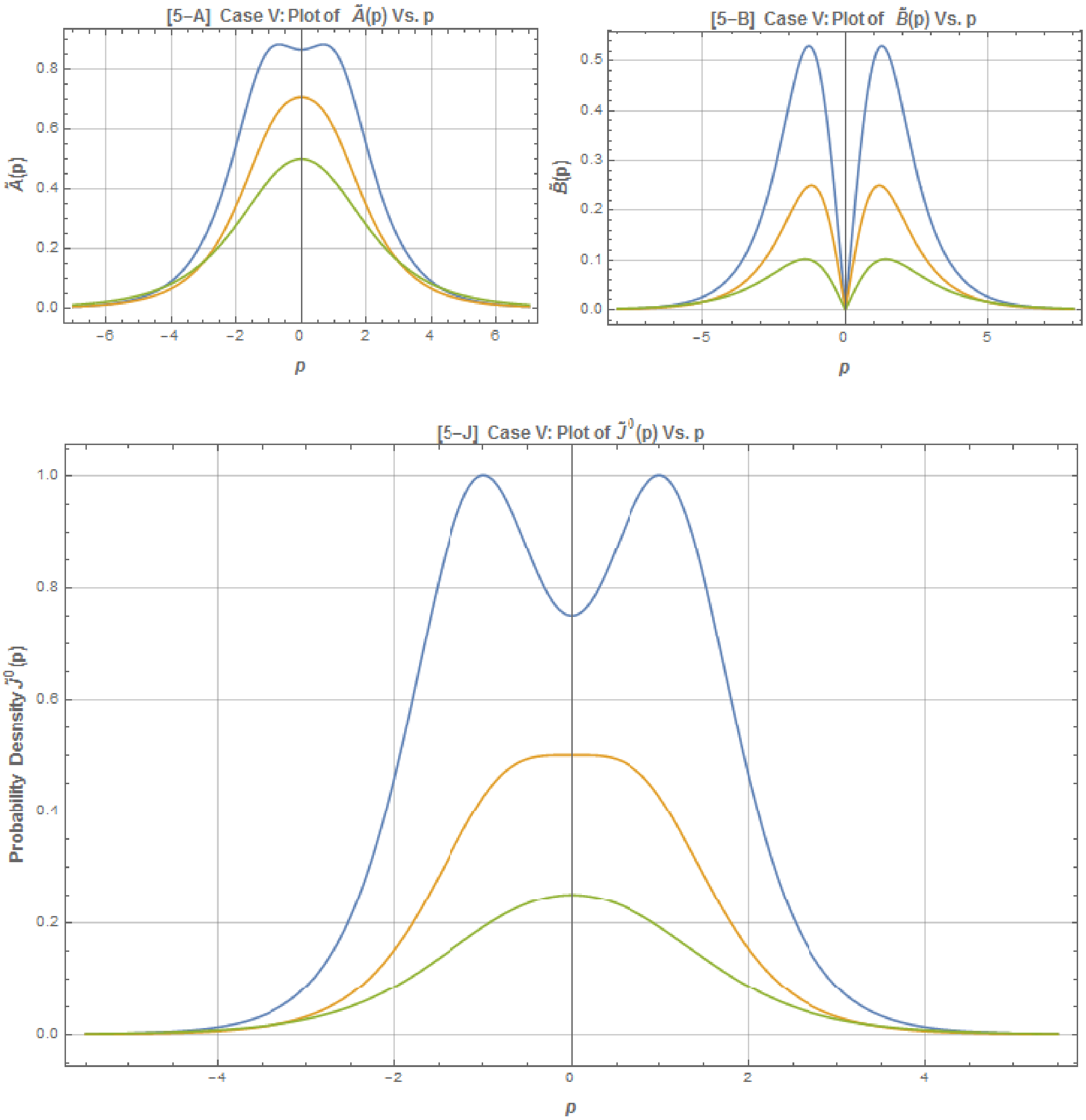

Figure 3: **Plots for Case [V]**. In all plots: [**Green**: case 5(a) with w=0.75], [**Orange**: case 5(a) with w=0.5], [**Blue**: case 5(b) with w=0.25]. Case 5(a) has global maxima at origin. Case 5(b) has local minima at origin and two maximas at two symmetrically opposite sides of origin at *non-zero p*. Both cases 5(a) and 5(b) are asymptotically vanishing.

For (w=2), we have plotted this solution in fig (5).

Following table summarizes all the cases:

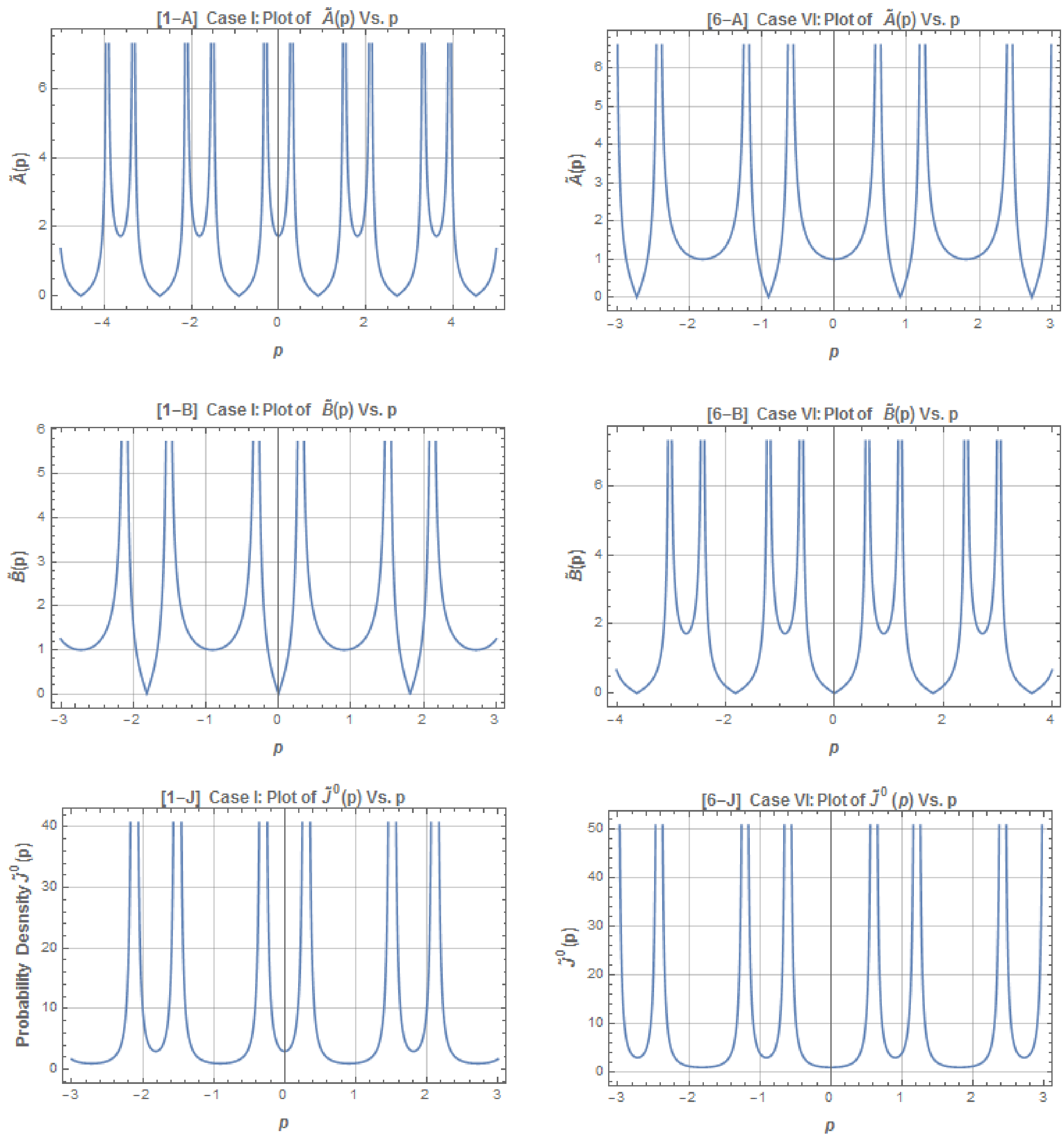

Figure 4: **Plots for case [I] and case [VI]**. The left column shows plots for case 1 with w=-2. The right column shows plots for case 6 with w= +2. Both the cases have unphysical solutions.

Cases	Solution(s) of linear (non-torsional) Dirac eqn	Solution(s) of non-linear Dirac eqn (with torsion)
Case I →	Physical (Plane wave)	Unphysical (having infinite singularities)
Case II →	Trivial solution	Trivial solution
Case III →	Unphysical (blows up exponentially at infinity)	Unphysical, (has 2 singularities)
Case IV →	Unphysical (blows up exponentially at infinity)	Unphysical (blows up exponentially at infinity)
Case V →	Unphysical (blows up exponentially at infinity)	Physical (No singularity)

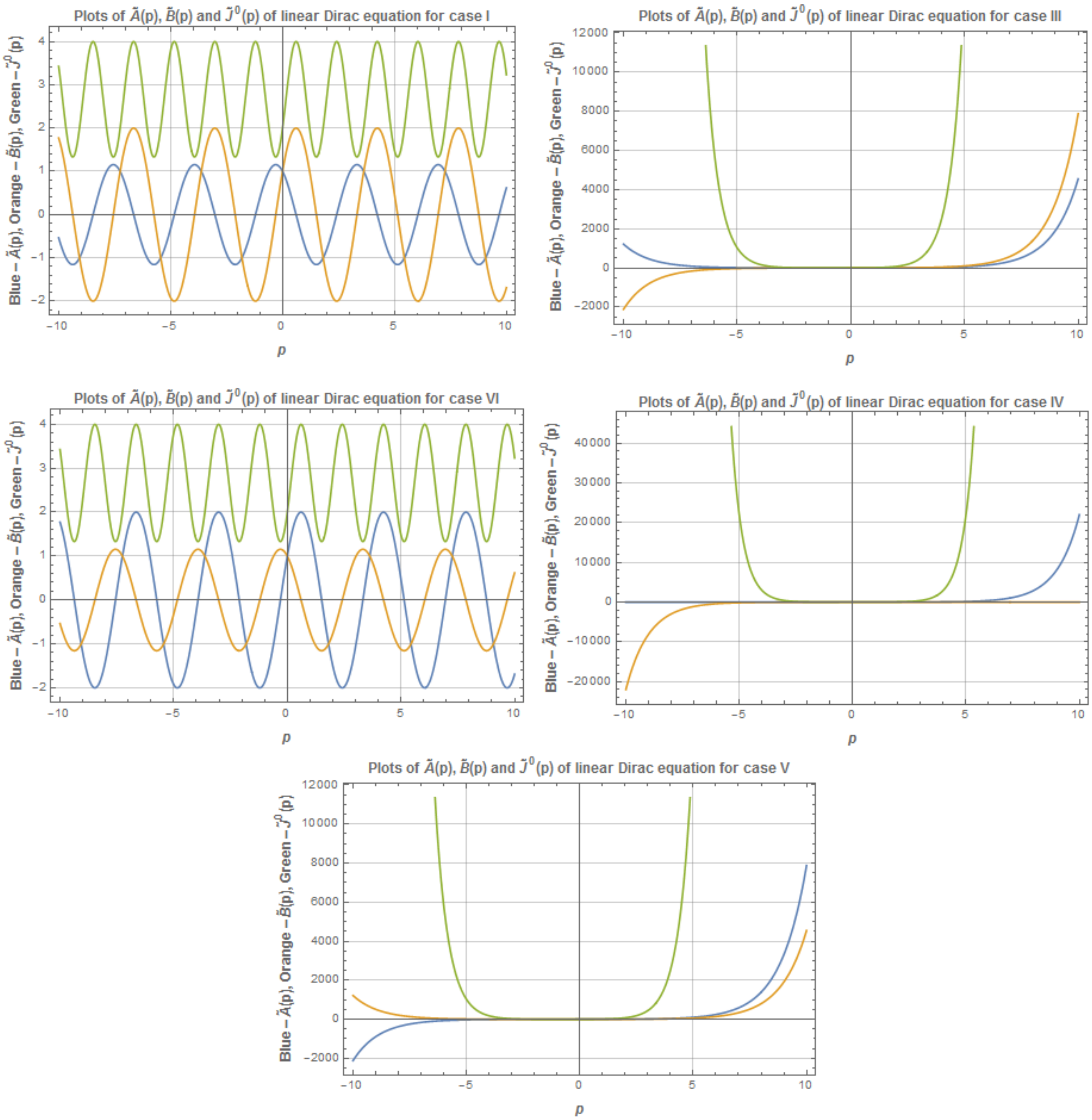

Figure 5: **Plots for solutions to linear (non-torsional) Dirac equation**. Only plane wave solutions (case 1,6) is physical. Rest all are unphysical.

6.2 Attempting plane wave solutions

We begin by substituting the following plane wave ansatz in (90 - 93) as follows:

$$\begin{bmatrix} F_1 \\ F_2 \\ G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} u^0 \\ u^1 \\ \bar{v}_{0'} \\ \bar{v}_{1'} \end{bmatrix} e^{ik.x} \tag{139}$$

With this ansatz, ξ and ξ^* are as follows:

$$\xi = u^A \bar{v}_{A'} \tag{140}$$
$$\xi^* = \bar{u}^{A'} v_A \tag{141}$$

We assume ξ to be a real constant such that $\xi = u^A \bar{v}_{A'} = \bar{u}^{A'} v_A = \xi^* =$ (real constant ξ). Putting the above ansatz in (90 - 93), we obtain:

$$(k_0 + k_3)u^0 + (k_1 + ik_2)u^1 - \mu(\xi)\bar{v}_{0'} = 0 \tag{142}$$
$$(k_0 - k_3)u^1 + (k_1 - ik_2)u^0 - \mu(\xi)\bar{v}_{1'} = 0 \tag{143}$$
$$(k_0 + k_3)\bar{v}_{1'} - (k_1 - ik_2)\bar{v}_{0'} - \mu(\xi)u^1 = 0 \tag{144}$$
$$(k_0 - k_3)\bar{v}_{0'} - (k_1 + ik_2)\bar{v}_{1'} - \mu(\xi)u^0 = 0 \tag{145}$$

where $\mu(\xi) = \sqrt{2}(b + a\xi)$. Note μ is a function of ξ which remains a undetermined quantity before finding a complete solution. We have to just make sure that ξ is a real constant.

$$\begin{pmatrix} (k_0+k_3) & (k_1+ik_2) & -\mu(\xi) & 0 \\ (k_1-ik_2) & (k_0-k_3) & 0 & -\mu(\xi) \\ 0 & -\mu(\xi) & -(k_1-ik_2) & (k_0+k_3) \\ -\mu(\xi) & 0 & (k_0-k_3) & -(k_1+ik_2) \end{pmatrix} \begin{pmatrix} u^0 \\ u^1 \\ \bar{v}_{0'} \\ \bar{v}_{1'} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{146}$$

We first assume $k_1 = k_2 = k_3 = 0$ (This is like attempting a solution in a rest frame). The above equation reduces to

$$\begin{pmatrix} k_0 & 0 & -\mu(\xi) & 0 \\ 0 & k_0 & 0 & -\mu(\xi) \\ 0 & -\mu(\xi) & 0 & k_0 \\ -\mu(\xi) & 0 & k_0 & 0 \end{pmatrix} \begin{pmatrix} u^0 \\ u^1 \\ \bar{v}_{0'} \\ \bar{v}_{1'} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{147}$$

For the above system to have solution, we must have det(coefficient matrix in 24) = 0. This gives

$$\Rightarrow [k_0^2 - \mu(\xi)^2]^2 = 0$$
$$\Rightarrow k_0 = \pm\mu(\xi)$$

6.2.1 Two cases for the plane wave solution(s)

Case I: $k_0 = +\mu(\xi)$: the general solution is of the form:

$$\begin{pmatrix} F_1 \\ F_2 \\ G_1 \\ G_2 \end{pmatrix} = \frac{\alpha_1}{\sqrt{V}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} e^{i\mu(\xi)x_0} + \frac{\beta_1}{\sqrt{V}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{i\mu(\xi)x_0} \tag{148}$$

where $|\alpha_1|^2 + |\beta_1|^2 = 1$ is the normalization condition.

Here, ξ and μ are as follows:

$$\xi = \frac{|\alpha_2|^2 + |\beta_2|^2}{V} = \frac{1}{V}$$
$$\mu = \sqrt{2}\left(b + \frac{a}{V}\right)$$

Case II: $k_0 = -\mu(\xi)$, general solution is of the form:

$$\begin{pmatrix} F_1 \\ F_2 \\ G_1 \\ G_2 \end{pmatrix} = \frac{\alpha_2}{\sqrt{V}} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} e^{-i\mu(\xi)x_0} + \frac{\beta_2}{\sqrt{V}} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{-i\mu(\xi)x_0} \tag{149}$$

where, $|\alpha_2|^2 + |\beta_2|^2 = 1$ is the normalization condition
Here ξ and μ are as follows:

$$\xi = \frac{-|\alpha_2|^2 - |\beta_2|^2}{V} = \frac{-1}{V}$$
$$\mu = \sqrt{2}\left(b - \frac{a}{V}\right)$$

What of the tensor $(T-S)_{\mu\nu}$? After explicit calculation (ref. Appendix C), we find that even in this case, $T-S$ never goes to zero for non-vanishing torsion.

6.3 Solution by reduction to (2+1) Dim in cylindrical coordinates (t,r,ϕ)

After assuming $\partial_z = 0$, the HD equations in cylindrical coordinates [94 - 97] are as follows:

$$r\partial_t F_1 + cr\partial_r F_2 e^{i\phi} + ic\partial_\phi F_2 e^{i\phi} F_1 = icr\sqrt{2}(b + a\xi)G_1 \tag{150}$$
$$r\partial_t F_2 + cr\partial_r F_1 e^{-i\phi} - ic\partial_\phi F_1 e^{-i\phi} = icr\sqrt{2}(b + a\xi)G_2 \tag{151}$$
$$r\partial_t G_2 - cr\partial_r G_1 e^{-i\phi} + ic\partial_\phi G_1 e^{-i\phi} = icr\sqrt{2}(b + a\xi^*)F_2 \tag{152}$$
$$r\partial_t G_1 - cr\partial_r G_2 e^{i\phi} - ic\partial_\phi G_2 e^{i\phi} = icr\sqrt{2}(b + a\xi^*)F_1 \tag{153}$$

We now take the ansatz, $F_2 = G_2$ and $F_1 = -G_1$

$$r\partial_t F_1 + r\partial_r F_2 e^{i\phi} + i\partial_\phi F_2 e^{i\phi} = -ir\sqrt{2}(b + a\xi)F_1 \tag{154}$$
$$r\partial_t F_2 + r\partial_r F_1 e^{-i\phi} - i\partial_\phi F_1 e^{-i\phi} = ir\sqrt{2}(b + a\xi)F_2 \tag{155}$$

We choose following ansatz in the above equation

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} iA(r)e^{\frac{i\phi}{2}} \\ B(r)e^{\frac{-i\phi}{2}} \end{bmatrix} e^{-i\omega t} \tag{156}$$

Putting this ansatz in above equations, we obtain the 2 differential equations as follows:

$$-rB\omega + r\partial_r A + \frac{A}{2} = r\sqrt{2}[b + a(B^2 - A^2)]B \tag{157}$$
$$rA\omega + r\partial_r B + \frac{B}{2} = r\sqrt{2}[b + a(B^2 - A^2)]A \tag{158}$$

We add and subtract the two equations above and make the following substitution:

$$\psi_1 = B(r) + A(r) \tag{159}$$
$$\psi_2 = B(r) - A(r) \tag{160}$$

To obtain:

$$-r\omega\psi_2 + r\psi_1' + \frac{\psi_1}{2} - r\sqrt{2}(b + a\psi_1\psi_2)\psi_1 = 0 \tag{161}$$

$$r\omega\psi_1 + r\psi_2' + \frac{\psi_2}{2} + r\sqrt{2}(b + a\psi_1\psi_2)\psi_2 = 0 \tag{162}$$

With $\omega = 0$, we have the solutions:

$$\psi_1 = \left[\frac{c_2 e^{\sqrt{2}br}}{r^{\left(\frac{1-2\sqrt{2}ac_1}{2}\right)}}\right] \qquad \psi_2 = \left[\frac{c_1 e^{-\sqrt{2}br} r^{\left(\frac{-1-2\sqrt{2}ac_1}{2}\right)}}{c_2}\right] \tag{163}$$

This is clearly unphysical because ψ_1 blows up $\forall$ non-zero c_2; and making c_2 zero blows up ψ_2. So, we conclude that, *static solution to the above system of equation is unphysical.* So ω can't be zero. Some further attempts to solve it numerically are in progress.

6.4 Solution by reduction to (3+1) Dim in spherical coordinates (t,r,θ,ϕ)

We begin by putting following ansatz in HD equations with spherical coordinates:

$$\begin{bmatrix} F_1 \\ F_2 \\ G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} R_{-\frac{1}{2}}(r)S_{-\frac{1}{2}}(\theta)e^{+i\phi/2} \\ R_{+\frac{1}{2}}(r)S_{+\frac{1}{2}}(\theta)e^{-i\phi/2} \\ R_{+\frac{1}{2}}(r)S_{-\frac{1}{2}}(\theta)e^{+i\phi/2} \\ R_{-\frac{1}{2}}(r)S_{+\frac{1}{2}}(\theta)e^{-i\phi/2} \end{bmatrix} e^{-i\omega t} \tag{164}$$

With this ansatz, (98) - (101) become:

$$\begin{aligned}&\left(-i\omega R_{-\frac{1}{2}}S_{-\frac{1}{2}} + \cos\theta R'_{-\frac{1}{2}}S_{-\frac{1}{2}} - \frac{\sin\theta}{r}R_{-\frac{1}{2}}S'_{-\frac{1}{2}} + \frac{1}{2r\sin\theta}R_{+\frac{1}{2}}S_{+\frac{1}{2}} + \sin\theta R'_{+\frac{1}{2}}S_{+\frac{1}{2}} + \frac{\cos\theta}{r}R_{+\frac{1}{2}}S'_{+\frac{1}{2}}\right)\\ &= i\sqrt{2}(b + a\xi)R_{+\frac{1}{2}}S_{-\frac{1}{2}}\end{aligned} \tag{165}$$

$$\begin{aligned}&\left(-i\omega R_{+\frac{1}{2}}S_{+\frac{1}{2}} - \cos\theta R'_{+\frac{1}{2}}S_{+\frac{1}{2}} + \frac{\sin\theta}{r}R_{+\frac{1}{2}}S'_{+\frac{1}{2}} - \frac{1}{2r\sin\theta}R_{-\frac{1}{2}}S_{-\frac{1}{2}} + \sin\theta R'_{-\frac{1}{2}}S_{-\frac{1}{2}} + \frac{\cos\theta}{r}R_{-\frac{1}{2}}S_{-\frac{1}{2}}\right)'\\ &= i\sqrt{2}(b + a\xi)R_{-\frac{1}{2}}(r)S_{+\frac{1}{2}}(\theta)\end{aligned} \tag{166}$$

$$\begin{aligned}&\left(-i\omega R_{-\frac{1}{2}}S_{+\frac{1}{2}} + \cos\theta R'_{-\frac{1}{2}}S_{+\frac{1}{2}} - \frac{\sin\theta}{r}R_{-\frac{1}{2}}S'_{+\frac{1}{2}} + \frac{1}{2r\sin\theta}R_{+\frac{1}{2}}S_{-\frac{1}{2}} - \sin\theta R'_{+\frac{1}{2}}S_{-\frac{1}{2}} - \frac{\cos\theta}{r}R_{+\frac{1}{2}}S'_{-\frac{1}{2}}\right)\\ &= i\sqrt{2}(b + a\xi^*)R_{+\frac{1}{2}}(r)S_{+\frac{1}{2}}(\theta)\end{aligned} \tag{167}$$

$$\begin{aligned}&\left(-i\omega R_{+\frac{1}{2}}(r)S_{-\frac{1}{2}}(\theta) - \cos\theta R'_{+\frac{1}{2}}S_{-\frac{1}{2}} + \frac{\sin\theta}{r}R_{+\frac{1}{2}}S'_{-\frac{1}{2}} - \frac{1}{2r\sin\theta}R_{-\frac{1}{2}}S_{+\frac{1}{2}} - \sin\theta R'_{-\frac{1}{2}}S_{+\frac{1}{2}} - \frac{\cos\theta}{r}R_{-\frac{1}{2}}S'_{+\frac{1}{2}}\right)\\ &= i\sqrt{2}(b + a\xi^*)R_{-\frac{1}{2}}S_{-\frac{1}{2}}\end{aligned} \tag{168}$$

Where

$$\xi = R_{-\frac{1}{2}}S_{-\frac{1}{2}}\bar{R}_{+\frac{1}{2}}\bar{S}_{-\frac{1}{2}} + R_{+\frac{1}{2}}S_{+\frac{1}{2}}\bar{R}_{-\frac{1}{2}}\bar{S}_{-\frac{1}{2}} \tag{169}$$

$$\xi^* = \bar{R}_{-\frac{1}{2}}\bar{S}_{-\frac{1}{2}}R_{+\frac{1}{2}}S_{-\frac{1}{2}} + \bar{R}_{+\frac{1}{2}}\bar{S}_{+\frac{1}{2}}R_{-\frac{1}{2}}S_{-\frac{1}{2}} \tag{170}$$

7 Discussion and outlook

In this paper, we formulated ECD theory in the NP formalism. The Dirac equation is carried to U_4 and presented (in NP) in (78 - 81). We have also provided a prescription for finding the covariant derivative on U_4, thereby allowing one to calculate objects like the generic EM tensor on U_4 etc. We have calculated the spin density term which acts as a correction to the dynamic (and symmetrical) EM tensor; the two of which contribute together to the Einstein tensor (made up of Christoffel connections). In addition, the contorsion spin coefficients in the NP formalism are also expressed in terms of the Dirac state.

We attempted finding solutions to HD equations on Minkowski space with torsion. Solutions after reducing the problem to (1+1) dimension in the variables (t, z) were found. This solution vanishes at infinity in the non-static case. However, static case is unphysical. We notice that, there is a singularity at finite z for positive energy solutions. In the case of vanishing torsion (where $a = 0$), a singularity exists at $z = 0$. It is interesting to notice that the addition of torsion shifts the singularity away from zero. Moreover, as the coupling length scale l decreases, the singularity shifts further away from zero, and never reaches zero for finite l. For negative energy solutions, there is no singularity and the function is physical as well as well-behaved – for any finite l, the solution is non-singular at zero.

Plane wave solutions were found in section (6.2). Next, we attempted finding solutions by reducing the problem to (2+1) dimensions in cylindrical coordinates with variables (t, r, ϕ). Static solutions to this were also found to be unphysical. However, Finding non-static solutions to (2+1) case (given in section 6.3) and the (3+1) case (given in section 6.4) is still under progress.

8 Appendices

8.1 Appendix A: Contorsion tensor ($K^{\mu\nu\alpha}$) components

Our aim is to write the contorsion tensor ($K^{\mu\nu\alpha}$) in the NP formalism eventually in terms of spinor components, with the contorsion tensor given by:

$$K^{\mu\nu\alpha} = -kS^{\mu\nu\alpha} = 2i\pi l^2 \bar{\psi}\gamma^{[\mu}\gamma^{\nu}\gamma^{\alpha]}\psi \tag{171}$$

Note, only four independent components of this tensor is excited by the Dirac field. Writing explicitly in the NP formalism, i.e., null tetrad basis, we have:

$$K_{(i)(j)(k)} = e_{(i)\mu}e_{(j)\nu}e_{(k)\alpha}K^{\mu\nu\alpha} \tag{172}$$

where $e_{(i)\mu} = (l_\mu, n_\mu, m_\mu, \bar{m}_\mu)$ for $i = 1, 2, 3, 4$ First, we consider the product $\gamma^\alpha\gamma^\beta\gamma^\mu$, defined as follows:

$$\gamma^\alpha\gamma^\beta\gamma^\mu = \begin{pmatrix} 0 & (\tilde{\sigma}^\alpha)^*(\sigma^\beta)^*(\tilde{\sigma}^\mu)^* \\ (\sigma^\alpha)^*(\tilde{\sigma}^\beta)^*(\sigma^\mu)^* & 0 \end{pmatrix} = 2\sqrt{2}\begin{pmatrix} 0_{2\times2} & K_{01} \\ K_{10} & 0_{2\times2} \end{pmatrix} \tag{173}$$

where, explicitly, expanding out the Van der Waarden symbols, we have:

$$K_{01} = \begin{bmatrix} +nln - n\bar{m}m - \bar{m}mn + \bar{m}nm & -nl\bar{m} + n\bar{m}l + \bar{m}m\bar{m} - \bar{m}nl \\ -mln + m\bar{m}m + lmn - lnm & +ml\bar{m} - m\bar{m}l - lm\bar{m} + lnl \end{bmatrix}^{\alpha\beta\mu} \tag{174}$$

$$K_{10} = \begin{bmatrix} +lnl - l\bar{m}m - \bar{m}ml + \bar{m}lm & +ln\bar{m} - l\bar{m}n - \bar{m}m\bar{m} + \bar{m}ln \\ +mnl - m\bar{m}m - nml + nlm & +mn\bar{m} - m\bar{m}n - nm\bar{m} + nln \end{bmatrix}^{\alpha\beta\mu} \tag{175}$$

With the expression for $\gamma^\alpha\gamma^\beta\gamma^\mu$, we can now define the world components of K. Next, we use (172) to calculate the contorsion spin coefficients[14] in the NP (null tetrad) basis. An an example, the solution for ρ_1 is given as:

$$\rho_1 = -K_{(1)(3)(4)} = -l_\mu m_\nu \bar{m}_\alpha K^{\mu\nu\alpha} = -2i\pi l^2 [l_\mu m_\nu \bar{m}_\alpha]\bar{\psi}\gamma^{[\mu}\gamma^{\nu}\gamma^{\alpha]}\psi \tag{176}$$

The only quantity giving a non-zero scalar product when contracted with $l_\mu m_\nu \bar{m}_\alpha$ is $n^\mu \bar{m}^\nu m^\alpha$ and corresponding permutations (given the definition of $\gamma^{[\mu}\gamma^{\nu}\gamma^{\alpha]}$), giving $l_\mu m_\nu \bar{m}_\alpha n^\mu \bar{m}^\nu m^\alpha = 1$. Thus:

$$\begin{aligned}
[l_\mu m_\nu \bar{m}_\alpha]\bar{\psi}\gamma^{[\mu}\gamma^{\nu}\gamma^{\alpha]}\psi &= \frac{\sqrt{2}}{3}\bar{\psi}\left(\begin{bmatrix} 0&0&-1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} - \begin{bmatrix} 0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} + \begin{bmatrix} 0&0&-1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} - \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&-1&0 \end{bmatrix} + \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0 \end{bmatrix} - \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&-1&0&0 \end{bmatrix} \right) \\
&= \frac{\sqrt{2}}{3}\begin{pmatrix} Q_0 & Q_1 & \bar{P}^{0'} & \bar{P}^{1'} \end{pmatrix} \begin{pmatrix} 0&0&-3&0\\0&0&0&0\\0&0&0&0\\0&3&0&0 \end{pmatrix} \begin{pmatrix} P^0 \\ P^1 \\ \bar{Q}_{0'} \\ \bar{Q}_{1'} \end{pmatrix} \\
&= \sqrt{2}(\bar{P}^{1'}P^1 - Q^1\bar{Q}^{1'})
\end{aligned}$$

This gives the full expression for ρ (redefining the spinor components as prescribed):

$$\rho = -K_{(1)(3)(4)} = -2\sqrt{2}i\pi l^2 [F_2\bar{F}_2 - G_1\bar{G}_1] \tag{177}$$

and similarly for the other spin coefficients.

8.2 Appendix B: The Dirac equation in U_4

The Dirac equation in U_4 (the *Hehl-Datta* equation) is given, in matrix form, as:

$$i\begin{pmatrix} 0 & (\tilde{\sigma}^\mu)^* \\ (\sigma^\mu)^* & 0 \end{pmatrix} \nabla_\mu \begin{pmatrix} P^A \\ \bar{Q}_{B'} \end{pmatrix} = \frac{1}{2\sqrt{2}l} \begin{pmatrix} P^A \\ \bar{Q}_{B'} \end{pmatrix} \tag{178}$$

Rewriting as a pair of matrix equations:

$$\begin{pmatrix} \sigma^\mu_{00'} & \sigma^\mu_{10'} \\ \sigma^\mu_{01'} & \sigma^\mu_{11'} \end{pmatrix} \nabla_\mu \begin{pmatrix} P^0 \\ P^1 \end{pmatrix} + \frac{i}{2\sqrt{2}l} \begin{pmatrix} -\bar{Q}^{1'} \\ \bar{Q}^{0'} \end{pmatrix} = 0 \tag{179}$$

$$\begin{pmatrix} \sigma^\mu_{11'} & -\sigma^\mu_{10'} \\ -\sigma^\mu_{01'} & \sigma^\mu_{00'} \end{pmatrix} \nabla_\mu \begin{pmatrix} -\bar{Q}^{1'} \\ \bar{Q}^{0'} \end{pmatrix} + \frac{i}{2\sqrt{2}l} \begin{pmatrix} P^0 \\ P^1 \end{pmatrix} = 0 \tag{180}$$

We will proceed to work through a solution for the first and third equation generated by this pair; the second and fourth follow along similar lines.

Equation 1:

$$\begin{aligned}
\frac{i}{2\sqrt{2}l}\bar{Q}^{1'} &= \sigma^\mu_{00'}\nabla_\mu P^0 + \sigma^\mu_{10'}\nabla_\mu P^1 \\
&= (\partial_{00'}P^0 + \Gamma^0{}_{i00'}P^i) + (\partial_{10'}P^1 + \Gamma^1{}_{i10'}P^i) \\
&= (D + \Gamma^0{}_{000'}P^0 + \Gamma^0{}_{100'}P^1) + (\delta^* + \Gamma^1{}_{010'}P^0 + \Gamma^1{}_{110'}P^1) \\
&= (D + \Gamma_{1000'} - \Gamma_{0010'})P^0 + (\delta^* + \Gamma_{1100'} - \Gamma_{0110'})P^1 \\
&= (D + \epsilon^o + \epsilon_1 - \rho^o - \rho_1)P^0 + (\delta^* + \pi^o + \pi_1 - \alpha^o - \alpha_1)P^1 \\
&= (D + \epsilon_0 - \rho_0)P^0 + (\delta^* + \pi_0 - \alpha_0)P^1 + \frac{3}{2}(\pi_1 P^1 - \rho_1 P^0)
\end{aligned} \tag{181}$$

Equation 3:

$$\begin{aligned}
\frac{i}{2\sqrt{2}l}P^0 &= -\sigma^\mu_{11'}\nabla_\mu \bar{Q}^{1'} - \sigma^\mu_{10'}\nabla_\mu \bar{Q}^{0'} + \frac{i}{2\sqrt{2}l}P^0 \\
&= -\bar{\sigma}^\mu_{11'}\nabla_\mu \bar{Q}^{1'} - \bar{\sigma}^\mu_{0'1}\nabla_\mu \bar{Q}^{0'} + \frac{i}{2\sqrt{2}l}P^0 \\
&= (\partial_{11'}\bar{Q}^{1'} + \bar{\Gamma}^{1'}{}_{i'1'1}\bar{Q}^{i'}) + (\partial_{10'}\bar{Q}^{0'} + \bar{\Gamma}^{0'}{}_{i'0'1}\bar{Q}^{i'}) \\
&= (\Delta\bar{Q}^{1'} + \bar{\Gamma}^{1'}{}_{0'1'1}\bar{Q}^{0'} + \bar{\Gamma}^{1'}{}_{1'1'1}\bar{Q}^{1'}) + (\delta^*\bar{Q}^{0'} + \bar{\Gamma}^{0'}{}_{0'0'1}\bar{Q}^{0'} + \bar{\Gamma}^{0'}{}_{1'0'1}\bar{Q}^{1'}) \\
&= (\Delta + \bar{\Gamma}_{1'1'0'1} - \bar{\Gamma}_{0'1'1'1})\bar{Q}^{1'} + (\delta^* + \bar{\Gamma}_{1'0'0'1} - \bar{\Gamma}_{0'0'1'1})\bar{Q}^{0'} \\
&= (\Delta + \mu^o + \mu_1 - \gamma^o - \gamma_1)\bar{Q}^{1'} + (\delta^* + \beta^o + \beta_1 - \tau^o - \tau_1)\bar{Q}^{0'} \\
&= (\Delta + \mu_0^* - \gamma_0^*)\bar{Q}^{1'} - (\delta^* + \beta_0^* - \tau_0^*)\bar{Q}^{0'} - \frac{3}{2}(\mu_1\bar{Q}^{1'} - \pi_1\bar{Q}^{0'})
\end{aligned} \tag{182}$$

where we have used the gamma matrices as defined in (46), computed the covariant derivatives using (48), (49) and the spin connections in terms of contorsion spin coefficients as given in (64). Using this procedure, the four Dirac equations in U_4 are obtained as:

$$(D + \epsilon_0 - \rho_0)F_1 + (\delta^* + \pi_0 - \alpha_0)F_2 + \frac{3}{2}(\pi_1 F_2 - \rho_1 F_1) = ib(l)G_1 \tag{183}$$

$$(\Delta + \mu_0 - \gamma_0)F_2 + (\delta + \beta_0 - \tau_0)F_1 + \frac{3}{2}(\mu_1 F_2 - \tau_1 F_1) = ib(l)G_2 \tag{184}$$

$$(D + \epsilon_0^* - \rho_0^*)G_2 - (\delta + \pi_0^* - \alpha_0^*)G_1 - \frac{3}{2}(\tau_1 G_1 - \rho_1 G_2) = ib(l)F_2 \tag{185}$$

$$(\Delta + \mu_0^* - \gamma_0^*)G_1 - (\delta^* + \beta_0^* - \tau_0^*)G_2 - \frac{3}{2}(\mu_1 G_1 - \pi_1 G_2) = ib(l)F_1 \tag{186}$$

where we have also redefined $\{P,Q\} \rightarrow \{F,G\}$, as per the substitution in (1) and to obtain a form that can be consistently compared with the primary source material in [6] (eqn. 108).

8.3 Appendix C: Calculating $(T-S)_{\mu\nu}$

In theories which consider a balance between the Riemannian and torsional curvatures (such as in [11], the tensor $(T-S)_{\mu\nu}$ is of paramount importance. Vanishing $(T-S)_{\mu\nu}$ would take the form of a 'balance condition', and represent a space with nonzero Riemannian curvature and torsion, but where the two exactly cancel each other out. The $(T-S)_{\mu\nu}$ tensor is defined as:

$$(T-S)_{\mu\nu} = T_{\mu\nu} - \frac{4\pi l^2}{\hbar c}\eta_{\mu\nu}S^{\alpha\beta\lambda}S_{\alpha\beta\lambda} \tag{187}$$

This tensor has 10 components. The 6 off-diagonal components are as follows:

$$\begin{aligned}(T-S)_{21} = \frac{i\hbar c}{4}\Big(&\bar{F}_1\partial_1 F_1 + \bar{F}_2\partial_1 F_2 + \bar{G}_1\partial_1 G_1 + \bar{G}_2\partial_1 G_2 - \bar{F}_2\partial_0 F_1 - \bar{F}_1\partial_0 F_2 + \bar{G}_2\partial_0 G_1 + \bar{G}_1\partial_0 G_2\\ &- \partial_1\bar{F}_1 F_1 - \partial_1\bar{F}_2 F_2 - \partial_1\bar{G}_1 G_1 - \partial_1\bar{G}_2 G_2 + \partial_0\bar{F}_2 F_1 + \partial_0\bar{F}_1 F_2 - \partial_0\bar{G}_2 G_1 - \partial_0\bar{G}_1 G_2\Big)\end{aligned} \tag{188}$$

$$\begin{aligned}(T-S)_{31} = \frac{i\hbar c}{4}\Big(&\bar{F}_1\partial_2 F_1 + \bar{F}_2\partial_2 F_2 + \bar{G}_1\partial_2 G_1 + \bar{G}_2\partial_2 G_2 + i\bar{F}_2\partial_0 F_1 - i\bar{F}_1\partial_0 F_2 - i\bar{G}_2\partial_0 G_1 + i\bar{G}_1\partial_0 G_2\\ &- \partial_2\bar{F}_1 F_1 - \partial_2\bar{F}_2 F_2 - G_1\partial_2\bar{G}_1 - \partial_2\bar{G}_2 G_2 - i\partial_0\bar{F}_2 F_1 + i\partial_0\bar{F}_1 F_2 + i\partial_0\bar{G}_2 G_1 - i\partial_0\bar{G}_1 G_2\Big)\end{aligned} \tag{189}$$

$$\begin{aligned}(T-S)_{41} = \frac{i\hbar c}{4}\Big(&\bar{F}_1\partial_3 F_1 + \bar{F}_2\partial_3 F_2 + \bar{G}_1\partial_3 G_1 + \bar{G}_2\partial_3 G_2 - \bar{F}_1\partial_0 F_1 + \bar{F}_2\partial_0 F_2 + \bar{G}_1\partial_0 G_1 - \bar{G}_2\partial_0 G_2\\ &- \partial_3\bar{F}_1 F_1 - \partial_3\bar{F}_2 F_2 - \partial_3\bar{G}_1 G_1 - \partial_3\bar{G}_2 G_2 + \partial_0\bar{F}_1 F_1 - \partial_0\bar{F}_2 F_2 - \partial_0\bar{G}_1 G_1 + \partial_0\bar{G}_2 G_2\Big)\end{aligned} \tag{190}$$

$$\begin{aligned}(T-S)_{32} = \frac{i\hbar c}{4}\Big(&i\bar{F}_2\partial_1 F_1 - i\bar{F}_1\partial_1 F_2 - i\bar{G}_2\partial_1 G_1 + i\bar{G}_1\partial_1 G_2 - \bar{F}_2\partial_2 F_1 - \bar{F}_1\partial_2 F_2 + \bar{G}_2\partial_2 G_1 + \bar{G}_1\partial_2 G_2\\ &- i\partial_1\bar{F}_2 F_1 + i\partial_1\bar{F}_1 F_2 + i\partial_1\bar{G}_2 G_1 - i\partial_1\bar{G}_1 G_2 + \partial_2\bar{F}_2 F_1 + \partial_2\bar{F}_1 F_2 - \partial_2\bar{G}_2 G_1 - \partial_2\bar{G}_1 G_2\Big)\end{aligned} \tag{191}$$

$$\begin{aligned}(T-S)_{42} = \frac{i\hbar c}{4}\Big(&-\bar{F}_1\partial_1 F_1 + \bar{F}_2\partial_1 F_2 + \bar{G}_1\partial_1 G_1 - \bar{G}_2\partial_1 G_2 - \bar{F}_2\partial_3 F_1 - \bar{F}_1\partial_3 F_2 + \bar{G}_2\partial_3 G_1 + \bar{G}_1\partial_3 G_2\\ &+ \partial_1\bar{F}_1 F_1 - \partial_1\bar{F}_2 F_2 - \partial_1\bar{G}_1 G_1 + \partial_1\bar{G}_2 G_2 + \partial_3\bar{F}_2 F_1 + \partial_3\bar{F}_1 F_2 - \partial_3\bar{G}_2 G_1 - \partial_3\bar{G}_1 G_2\Big)\end{aligned} \tag{192}$$

$$\begin{aligned}(T-S)_{43}(\{\}) = \frac{i\hbar c}{4}\Big(&-\bar{F}_1\partial_2 F_1 + \bar{F}_2\partial_2 F_2 + \bar{G}_1\partial_2 G_1 - \bar{G}_2\partial_2 G_2 + i\bar{F}_2\partial_3 F_1 - i\bar{F}_1\partial_3 F_2 - i\bar{G}_2\partial_3 G_1 + i\bar{G}_1\partial_3 G_2\\ &+ \partial_2\bar{F}_1 F_1 - \partial_2\bar{F}_2 F_2 - \partial_2\bar{G}_1 G_1 + \partial_2\bar{G}_2 G_2 - i\partial_3\bar{F}_2 F_1 + i\partial_3\bar{F}_1 F_2 + i\partial_3\bar{G}_2 G_1 - i\partial_3\bar{G}_1 G_2\Big)\end{aligned} \tag{193}$$

The diagonal components are as follows:

$$(T-S)_{11} = \frac{i\hbar c}{2}\Big(\bar{G}_1\partial_0 G_1 + \bar{G}_2\partial_0 G_2 - \partial_0\bar{G}_1 G_1 - \partial_0\bar{G}_2 G_2 + \bar{F}_1\partial_0 F_1 + \bar{F}_2\partial_0 F_2 - \partial_0\bar{F}_1 F_1 - \partial_0\bar{F}_2 F_2\Big) - 6\pi\hbar c l^2 \xi\xi^* \tag{194}$$

$$(T-S)_{22} = \frac{i\hbar c}{2}\Big(- \bar{F}_2\partial_1 F_1 - \bar{F}_1\partial_1 F_2 + \bar{G}_2\partial_1 G_1 + \bar{G}_1\partial_1 G_2 + \partial_1\bar{F}_2 F_1 + \partial_1\bar{F}_1 F_2 - \partial_1\bar{G}_2 G_1 - \partial_1\bar{G}_1 G_2\Big) + 6\pi\hbar c l^2 \xi\xi^* \tag{195}$$

$$(T-S)_{33} = \frac{i\hbar c}{2}\Big(i\bar{F}_2\partial_2 F_1 - i\bar{F}_1\partial_2 F_2 - i\bar{G}_2\partial_2 G_1 + i\bar{G}_1\partial_2 G_2 - i\partial_2\bar{F}_2 F_1 + i\partial_2\bar{F}_1 F_2 + i\partial_2\bar{G}_2 G_1 - i\partial_2\bar{G}_1 G_2\Big) + 6\pi\hbar c l^2 \xi\xi^* \tag{196}$$

$$(T-S)_{44} = \frac{i\hbar c}{2}\Big(- \bar{F}_1\partial_3 F_1 + \bar{F}_2\partial_3 F_2 + \bar{G}_1\partial_3 G_1 - \bar{G}_2\partial_3 G_2 + \partial_3\bar{F}_1 F_1 - \partial_3\bar{F}_2 F_2 - \partial_3\bar{G}_1 G_1 + \partial_3\bar{G}_2 G_2\Big) + 6\pi\hbar c l^2 \xi\xi^* \tag{197}$$

We can now calculate this tensor for the various solutions to the HD equations on Minkowski space with torsion, to probe the feasibility of this balance condition.

$(T-S)_{\mu\nu}$ for non-static Solution 1+1 dimension (t,z)

$$(T-S)_{\mu\nu} = \hbar c \begin{pmatrix} \left(\Lambda[A^2+B^2] - \frac{a[A^2-B^2]^2}{2\sqrt{2}}\right) & 0 & -\Lambda AB & 0 \\ 0 & \left(\frac{a[A^2-B^2]^2}{2\sqrt{2}}\right) & 0 & 0 \\ -\Lambda AB & 0 & \left(\frac{a[A^2-B^2]^2}{2\sqrt{2}}\right) & 0 \\ 0 & 0 & 0 & \left([AB'-BA'] + \frac{a[A^2-B^2]^2}{2\sqrt{2}}\right) \end{pmatrix} \tag{198}$$

Λ is a free parameter in the solution. We will analyze this tensor "T-S" for various types of values of Λ.

$(T-S)_{\mu\nu}$ for Plane wave solutions

For case I:

$$(T-S)_{\mu\nu} = \hbar c \begin{pmatrix} -\left(\frac{V+18\pi l^3}{V^2 l}\right) & 0 & 0 & 0 \\ 0 & \left(\frac{6\pi l^2}{V^2}\right) & 0 & 0 \\ 0 & 0 & \left(\frac{6\pi l^2}{V^2}\right) & 0 \\ 0 & 0 & 0 & \left(\frac{6\pi l^2}{V^2}\right) \end{pmatrix} \tag{199}$$

For case II

$$(T-S)_{\mu\nu} = \hbar c \begin{pmatrix} -\left(\frac{V-18\pi l^3}{V^2 l}\right) & 0 & 0 & 0 \\ 0 & \left(\frac{6\pi l^2}{V^2}\right) & 0 & 0 \\ 0 & 0 & \left(\frac{6\pi l^2}{V^2}\right) & 0 \\ 0 & 0 & 0 & \left(\frac{6\pi l^2}{V^2}\right) \end{pmatrix} \tag{200}$$

Comments:

We observe that for both cases that $(T-S)_{\mu\nu}$ goes to zero only when $V \longrightarrow \infty$. But V going to ∞ implies ξ going to zero. So in case of vanishing torsion only, $(T-S)$ has any hopes of becoming zero.

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