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Internship Report

The Geroch group and holography

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Abstract

We present various solution generating algorithms for spacetimes with a Killing field using dimensional reduction. In particular, we discuss Geroch's algorithm in Ricci-flat spacetime and its extension to Einstein spaces (locally anti-de Sitter), where only a subgroup of the original $SL(2,\mathbb{R})$ Ehlers-Geroch group generates new solutions. This subgroup does not contain the celebrated Schwarzschild-TaubNUT-AdS solution, which is known to exist. Holography provides an alternative "dimensionally reduced" solution generating algorithm which contains this missing subgroup, dubbed $\mathcal{U}$-duality group. This motivated us to study the holographic dual of the Geroch group and we present here our attempts to do this for the Ricci-flat spacetimes (in future, we would also like to do the same for the AdS case and compare the dual with already found $\mathcal{U}$-duality group).

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Introduction

Einstein's field equations, are a set of coupled non-linear partial differential equations and finding exact solutions to these is a very difficult task. Most known exact solutions heavily rely on symmetries of the spacetime and the only physically interesting ones that we know are perfect fluids, electrovacuum, scalar field or point mass solutions (and their combinations). Studying and finding exact solutions is crucial as even the 'nonphysical' exact solutions like Taub-NUT have been known to provide valuable references to compare approximate or numerical results to check the validity of approximation techniques [1, 2] (see section 1.3 for an example). This is why solution generating techniques like the ones developed by Ehlers [3] are so crucial.

In his seminal papers in 1970 [4, 5], Geroch generalized Ehlers' algorithm [6, 7] for generating vacuum solutions of Einstein's equations in Ricci-flat spacetimes with a Killing field. Given a Riemannian manifold $\mathcal{M}$ with a Ricci flat metric g possessing Killing direction ξ, there exists a one-to-one correspondence between $(\mathcal{M}, g, \xi)$ and $(\mathcal{S}, h, \lambda, \omega)$, where λ and ω are the norm and scalar twist of the Killing field ξ and h is the metric on $\mathcal{S}$, which is generated by quotienting $\mathcal{M}$ by the orbits of ξ. It was then shown that the dynamics of g on $\mathcal{M}$ is equivalent to the dynamics of (h, λ, ω) in a 3d sigma model. This sigma model is invariant under $SL(2, \mathbb{R})$ transformations, which can be used to find a new triplet (h', λ', ω') that obey the same dynamical equations. These triplets can then be promoted to find new four-dimensional vacuum solutions g' with one Killing field ξ'. An example of this is Schwarzschild to Taub-NUT which is outlined in section 1.3.

In [8, 9], this technique was extended to Einstein spaces with non-vanishing cosmological constant. It was shown that in this case, only a subgroup of $SL(2, \mathbb{R})$ generates new solutions. Unfortunately, this extension could not answer if it is possible to get Taub-NUT-AdS from Schwarzschild-AdS using this Geroch-like process, both of which are known to exist independently. In a separate study in [10], it was shown that a holography based solution generating algorithm contains this missing subgroup for the AdS case. This motivates us to ask: 1. *How is this holographic solution generating algorithm related to the holographic duals of* $(\mathcal{M}, g)$ *and* $(\mathcal{M}, g')$? Can holography also explain why the algorithm generated in [10] failed to account for the complete Geroch group, i.e., does holography have a handle over the hidden symmetries of the bulk? and 2. *Can holography provide new insights for the Ricci-flat case* too?

The goal of this report is to pave the way to answering these two questions. In [9–13], it was observed that a derivative expansion gauge like the one used for fluid/gravity correspondence is better suited for the task at hand since the bulk metric can be written in a closed-form. Derivative expansion also admits a consistent zero-cosmological constant limit. Indeed, in [14–16] it was also shown that for flat spacetimes, the zero-cosmological limit in the bulk corresponds to "Carrollian" limit on the boundary, where Carrollian algebra is the vanishing speed of light contraction of the Poincare algebra (as opposed to the Galilean algebra which is infinite speed of light contraction of Poincare algebra).

In this report, I will first review Geroch's algorithm and demonstrate its use through an example in chapter 1. In chapter 2, I will outline the procedure that was proposed in [8] to extend Geroch's algorithm to AdS and check its range of validity. Chapter 3 will deal with the holographic solution generating algorithm of [10] which contains the missing subgroup and apply it to the most general family of Petrov type-D solutions, the Plebański-Demiański Einstein spaces. Chapter 4 will deal with flat holography where in section 4.3 we outline our attempts to find the Geroch dual. Appendices A to I will provide complimentary material and examples for interested readers.

Chapter 1

Geroch in Ricci flat spaces

In this chapter we briefly review the original Geroch's solution generating algorithm for Ricci flat spacetimes of [4, 5]. We demonstrate it through an example by generating Taub-NUT solution, starting from the Schwarzschild solution. The appendix of [4] provides a very neat complementary reading on the contents of this chapter.

1.1 From 4D to 3D and back

Consider a Ricci flat vacuum solution of Einstein's equations $(\mathcal{M}, g)$ with a Killing field ξ^a that is everywhere spacelike or timelike. We define its norm and twist as

$$\begin{aligned} \lambda &= \xi^a \xi_a, \\ \omega_a &= \epsilon_{abcd}\xi^b \nabla^c \xi^d = -2 i_\xi \star_g^4 \mathrm{d}\xi, \end{aligned} \tag{1.1}$$

where i_ξ represents contraction with ξ and $\star_g^n$ is the n-dimensional Hodge duality[1] with respect to metric g. For Ricci-flat spacetimes[2], we locally define scalar twist ω, as

$$\omega_a = \mathrm{d}\omega. \tag{1.2}$$

Since the metric g admits a Killing vector, one can define the "quotient" of $\mathcal{M}$ with respect to the action of one-parameter group generated by ξ as a three dimensional space S. Because there exists a one-to-one correspondence between tensor fields on $\mathcal{S}$ and certain tensor fields on $\mathcal{M}$[3], we can equip $\mathcal{S}$ with a manifold structure, with metric h defined as

$$h_{ab} = g_{ab} - \frac{\xi_a \xi_b}{\lambda}. \tag{1.3}$$

Using this one-to-one correspondence, dynamics of Einstein's equations in presence of a Killing vector in $(\mathcal{M}, g)$ can be recast in terms of dynamics of (h, ω, λ) on S, i.e.,

$$\begin{aligned} \tilde{\mathcal{R}}_{ab} &= -2(\tau - \tilde{\tau})^{-1} \tilde{\mathcal{D}}_{(a}\tau \tilde{\mathcal{D}}_{b)}\tilde{\tau}, \\ \tilde{\mathcal{D}}^2 \tau &= 2(\tau - \tilde{\tau})^{-1} \tilde{\mathcal{D}}_m \tau \tilde{\mathcal{D}}_n \tau \tilde{h}^{mn}, \end{aligned} \tag{1.4}$$

where $\tau = \omega + i\lambda$ and $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{R}}_{ab}$ are the Levi-Civita covariant derivative and Ricci tensor with respect to $\tilde{h}_{ab} = \lambda h_{ab}$. Equation (1.4) in principle provides new solutions (h', ω', λ') such that we can define a new four dimensional metric g' on $\mathcal{M}$ with Killing vector $\xi' = \eta\lambda'$ (normalized such that $i_\xi \eta = 1$) as

$$g'_{ab} = h'_{ab} + \frac{\xi'_a \xi'_b}{\lambda'}. \tag{1.5}$$

[1] Hodge star operator on n-dimensional manifold maps a p-form to (n-p) form

$$(\star A)_{\mu_1..\mu_{n-p}} = \frac{1}{p!}\eta^{\nu_1..\nu_p}{}_{\mu_1..\mu_{n-p}} A_{\nu_1..\nu_p}$$

where $\eta_{\mu_1\mu_2...} = \sqrt{-g}\epsilon_{\mu_1\mu_2...}$

[2] Using Cartan's structure equations, $\star d \star d\xi = 2 i_\xi Ric$, where Ric stands for Ricci tensor. Hence for Ricci-flat spacetimes $\star d\xi$ is a closed form (and hence locally exact)

[3] It was shown in [4] that the entire tensor algebra on S is completely and uniquely mirrored by tensors fields T on $\mathcal{M}$ which satisfy $i_\xi T = 0$ and $\mathcal{L}_\xi T = 0$. See appendix of [4] for more details.

Here we have defined η using the fact that for a curl free skew field F' on S, equations (1.4) imply that the pull back F_{ab} on $\mathcal{M}$ is closed, i.e.,

$$F'_{ab} := \frac{1}{(-\lambda')^{3/2}} \star^3_{h'} \mathrm{d}\omega' = \mathrm{d}\eta' \tag{1.6}$$

1.2 Group structure

The equations (1.4) are invariant under $SL(2,\mathbb{R})$ transformations, i.e., under

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d}, \tag{1.7}$$

where the $a, b, c, d \in \mathbb{R}$ satisfy $ad - bc = 1$. This is what is usally refer to as the Geroch group[4] This invariance allows us to construct new solutions on $\mathcal{S}$, (h', τ') from an initial solution (h, τ). However, not all transformations lead to new solutions. Indeed, transformations with $c = 0$ only leads to conformal rescalings $h'_{ab} = d^2 h_{ab}, \lambda' = d^{-2}\lambda, \omega' = d^{-1}(a\omega + b)$ and are related to diffeomorphisms on $\mathcal{M}$.

In section 2.1, we will show that the dynamical equations (1.4) for pure Einstein gravity correspond to a non-linear σ-model with a non-compact symmetric space $SL(2,\mathbb{R})/SO(1,1)$. The non-trivial transformations in (1.7), i.e., $(h', \tau') \to (h, \tau)$ then corresponds to transforming the sigma-model structure from $SL(2,\mathbb{R})/SO(1,1)$ to $SL(2,\mathbb{R})/SO(2)$ [17]. For the Schwarzschild to TaubNUT case discussed in the next section, these non-trivial transformations correspond to rotation in the mass, m and NUT parameter, n plane: $\mu + i\nu \to e^{-i\chi}(\mu + i\nu)$.

1.3 Schwarzschild and Taub-NUT geometries

To demonstrate the power of this technique, we will work out how one can get Taub-NUT geometries from Schwarzschild geometries. A brief overview of Taub-NUT geometries can be found in Appendix A. Starting with the Schwarzschild geometry,

$$g = -\left(1 - \frac{\mu}{r}\right) \mathrm{d}t \otimes \mathrm{d}t + \left(1 - \frac{\mu}{r}\right)^{-1} \mathrm{d}r \otimes \mathrm{d}r + r^2 \left(\mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta \; \mathrm{d}\phi \otimes \mathrm{d}\phi\right), \tag{1.8}$$

and choose ξ^α to be the timelike Killing vector ∂_t, i.e., $\xi^\alpha_t = (1, 0, 0, 0)$. We use equation (1.1) to find the norm and scalar twist of this Killing field as:

$$\begin{aligned} \lambda = g_{tt} &= -\left(1 - \frac{\mu}{r}\right), \\ \omega &= 0. \end{aligned} \tag{1.9}$$

Using (1.3), the metric on S is given by

$$h = \left(1 - \frac{\mu}{r}\right)^{-1} \mathrm{d}r \otimes \mathrm{d}r + r^2 \left(\mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta \; \mathrm{d}\phi \otimes \mathrm{d}\phi\right). \tag{1.10}$$

The form of (1.4) suggests that if we take $\tilde{h}'(= \lambda' h')$ to be equal to $\tilde{h}(= \lambda h)$, then τ' is related to τ by an element of $SL(2,\mathbb{R})$, through (1.7). Choosing to parametrize it as,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix} \tag{1.11}$$

[4] The precise mathematical definition of the Geroch group is given in Appendix B.

we get

$$\begin{aligned}\lambda' &= \frac{\lambda}{\cos^2\chi + \lambda^2 \sin^2\chi},\\ \omega' &= \frac{\sin\chi\cos\chi(1-\lambda^2)}{\cos^2\chi + \lambda^2\sin^2\chi},\\ h'_{ab} &= (\cos^2\chi + \lambda^2\sin^2\chi)h_{ab}.\end{aligned} \tag{1.12}$$

Finally using equation(1.6), we get that the field η' is such that

$$\nabla_{[a}\eta'_{b]} = \frac{1}{(-\lambda')^{3/2}}\tilde{h}^{cd}\epsilon_{abc}\partial_d\omega'. \tag{1.13}$$

This equation can be solved componentwise to give,

$$\begin{aligned}\nabla_{[t}\eta'_{i]} &= 0,\\ \nabla_{[r}\eta'_{\theta]} &= 0,\\ \nabla_{[r}\eta'_{\phi]} &= -\frac{1}{(-\lambda')^{3/2}}h'^{\theta\theta}\sqrt{-h'}\partial_\theta\omega' = 0,\\ \nabla_{[\theta}\eta'_{\phi]} &= \frac{1}{(-\lambda')^{3/2}}h'^{rr}\sqrt{-h'}\partial_r\omega',\end{aligned} \tag{1.14}$$

where $i = (r, \theta, \phi)$ and $\epsilon_{abc} = \frac{1}{\sqrt{\pm\lambda}}\epsilon_{abcd}\xi^d$. Choosing an ansatz for η',

$$\eta'_\alpha = (1, 0, 0, a(\theta)), \tag{1.15}$$

such that $i_\xi\eta' = 1$, and F is skew symmetric and substituting it in (1.14), we get

$$a(\theta) = -2\mu\cos\theta\sin 2\chi. \tag{1.16}$$

Hence the new metric g' of equation (1.5) from Schwarzschild geometry is

$$\begin{aligned} g' &= \lambda'\mathrm{d}t\otimes\mathrm{d}t - \frac{1}{\lambda'}\mathrm{d}r\otimes\mathrm{d}r + \frac{\lambda}{\lambda'}r^2\mathrm{d}\theta\otimes\mathrm{d}\theta + \lambda'\left(2a\mathrm{d}t + \left(\frac{\lambda}{\lambda'^2}r^2\sin^2\theta + a^2\right)\mathrm{d}\phi\right)\otimes\mathrm{d}\phi,\\ &= \lambda'\left(\mathrm{d}t + a\mathrm{d}\phi\right)^2 - \frac{1}{\lambda'}\mathrm{d}r^2 + \frac{\lambda}{\lambda'}r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right).\end{aligned} \tag{1.17}$$

Clearly if we choose

$$\sin 2\chi = \frac{\nu}{\mu}, \tag{1.18}$$

then (1.17) is a Taub-NUT metric (A.1) with ν as the NUT parameter. Hence, as we claimed in section 1.2, the Geroch group rotates the NUT and the mass parameter by an angle 2χ. Through this easy example we have demonstrated the strength of Geroch's algorithm by generating Taub-NUT geometries starting from Schwarzschild metric. Also as stated previously, not all Killing vectors ξ lead to a new geometry. Indeed had we chosen the spacelike Killing vector ∂_ϕ instead, we would have obtained a metric that was diffeomorphic to Schwarzschild metric. In fact, only non-null Killings lead to non-trivial solutions. To the best of my knowledge, there is no explanation available for this in the literature. However, by the end of this report, we will be able to give a physical motivation for why one should have expected this.

Our next goal is extend this algorithm to AdS spacetimes for Einstein spaces and to understand holographic equivalent of this algorithm.

Chapter 2

Geroch in AdS

In chapter 1 we insisted on working with new configurations (ω', λ', h') that keep $\tilde{h}' = \tilde{h}$, where $\tilde{h} = \lambda h$. However, for Einstein spaces[1] this restriction leads to the space of solutions that cannot simultaneously accommodate Schwarzschild-AdS and Taub-NUT-AdS spaces (see for example the last two paragraphs of section 2.2). In [8], the authors suggested that this obstruction can be overcome by demanding that h' is in the conformal class of h, without any further restrictions. To work with the same machinery as chapter 1, the authors introduced an extra scalar field κ as the conformal factor and showed that the dynamics of $(\kappa, \omega, \lambda)$ is captured by a three-dimensional sigma model with target space conformal to $\mathbb{R} \times H_2$, where H_2 is Lobatchevski plane. This dynamical scalar field is useful for probing mini-superspace of solutions with a frozen reference metric.

In this chapter I will briefly introduce the Geroch group on AdS Einstein spacetimes and outline the non-linear σ-model structure of the reduced spacetime discussed in [8]. Finally we also discuss the issues of integrability with Geroch group on Einstein spaces and introduce the holographic $\mathcal{U}$-duality [9].

2.1 Dimensional reduction and Sigma model action

The power of Geroch's algorithm lies in the observation that each given stationary, axially symmetric solution is accompanied by an infinite family of potentials, which in turn allowed for an infinite parameter set of infinitesimal transformations acting on the initial solution [17]. Quite naturally the field equations for such stationary, axially symmetric solutions of the 4-dimensional theory can be dimensionally reduced, which leads to non-linear σ-model for a non-compact symmetric space G/H [18]. Let us briefly outline why this is the case as well as the field equations corresponding to the model:

Consider a non-linear σ-model coupled to gravity, i.e., fields with values in target space $\bar{\Phi}$ with coordinates $\bar{\phi}^i$ and target metric $\bar{\gamma}_{ij}$ over a spacetime manifold Σ with coordinates x^α and metric $g_{\alpha\beta}(x)$. The action and field equations for it are:

$$\begin{aligned} S_{\bar{\Phi}} = \int_\Sigma \sqrt{g}\mathrm{d}x \left(-\frac{1}{2}R(x) + \frac{1}{2}g^{\alpha\beta}(x)\partial_\alpha\bar{\phi}^i(x)\partial_\beta\bar{\phi}^j(x)\bar{\gamma}_{ij}(\bar{\phi}(x))\right) \\ R_{\alpha\beta} - \partial_\alpha\bar{\phi}^i(x)\partial_\beta\bar{\phi}^j(x)\bar{\gamma}_{ij}(\bar{\phi}(x)) = 0 \\ \mathcal{D}^\alpha\partial_\alpha\bar{\phi}^i(x) = 0 \end{aligned} \tag{2.1}$$

In the case when $\bar{\Phi}$ is the non-compact Riemannian symmetric space $\bar{G}/\bar{H}$, the sigma model can be written as (see B.3),

$$\begin{aligned} S_{\bar{G}/\bar{H}} = \int_\Sigma \sqrt{g}\mathrm{d}x \left(-\frac{1}{2}R(x) + \frac{1}{2}g^{\alpha\beta}(x)\langle\bar{J}_\alpha, \bar{J}_\beta\rangle\right), \\ R_{\alpha\beta} - (x)\langle\bar{J}_\alpha, \bar{J}_\beta\rangle = 0, \\ \mathcal{D}^\alpha\bar{J}_\alpha = 0. \end{aligned} \tag{2.2}$$

[1]Reader is reminded that for Einstein spaces, $R_{ab} \propto g_{ab}$

where $\bar{J} = \frac{1}{2}\bar{M}^{-1}\partial\bar{M}$ is the current corresponding to the "metric" M[2]. For the Geroch group, M can be chosen to be of the form of equation (C.4), with Kaluza-Klein fields (λ, ω_a).

It is worth mentioning that for Geroch group, the collection of effective transformations are $SL(2,\mathbb{R})/N$, where N represents gauge transformations (in terms of the three dimensional Lorentz group, $O(2,1)$, N is the subgroup of null rotations [4]). Unfortunately N is not a normal subgroup of $SL(2,\mathbb{R})$ and hence G/H is not a group and has no natural group structure. This is why we had to impose an artificial group structure on it in section 1.2 by choosing the representative to be of the form of equation (1.11), which is only a subgroup of $SL(2,\mathbb{R})$ and intersects each coset of N exactly once.

2.2 Extension of Geroch to AdS spacetimes

Assuming that $\hat{h}$ is some three-dimensional reference metric, the condition for h' to be in the same conformal class as h can be reformulated as

$$h_{ab} = \frac{\kappa}{\lambda}\hat{h}_{ab}, \tag{2.3}$$

which allows κ, a dilaton-like field that captures one of the degrees of freedom of $\hat{h}$, to remain dynamical together with the Kaluza-Klein vector and scalar fields (ω_a, λ). Equations (1.4), then read

$$\begin{aligned}
\hat{\mathcal{R}}_{ab} &= -2(\tau-\tilde{\tau})^{-1}\hat{\mathcal{D}}_{(a}\tau\hat{\mathcal{D}}_{b)}\tilde{\tau} + \frac{1}{2\kappa}\left(\hat{\mathcal{D}}_a\hat{\mathcal{D}}_b\kappa + \hat{h}_{ab}\hat{\mathcal{D}}^c\hat{\mathcal{D}}_c\kappa\right) \\
&\quad - \frac{1}{4\kappa^2}\left(3\hat{\mathcal{D}}_a\kappa\hat{\mathcal{D}}_b\kappa + \hat{h}_{ab}\hat{\mathcal{D}}^c\kappa\hat{\mathcal{D}}_c\kappa\right) + 4i\Lambda\frac{\kappa}{\tau-\tilde{\tau}}\hat{h}_{ab}, \\
\hat{\mathcal{D}}^2\tau &= 2(\tau-\tilde{\tau})^{-1}\hat{\mathcal{D}}_m\tau\hat{\mathcal{D}}_n\tau\hat{h}^{mn} - \frac{1}{2\kappa}\hat{\mathcal{D}}_m\kappa\hat{\mathcal{D}}_n\tau\hat{h}^{mn} - 2i\Lambda\kappa, \\
\hat{\mathcal{D}}^2\kappa &= \frac{3}{4\kappa}\hat{\mathcal{D}}_m\kappa\hat{\mathcal{D}}_n\kappa\hat{h}^{mn} + \frac{\kappa}{(\tau-\tilde{\tau})^2}\hat{\mathcal{D}}_m\tau\hat{\mathcal{D}}_n\tilde{\tau}\hat{h}^{mn} - 6i\Lambda\frac{\kappa^2}{\tau-\tilde{\tau}} + \frac{\kappa}{2}\hat{\mathcal{R}}.
\end{aligned} \tag{2.4}$$

These equations are the equivalent of field equations (2.2) if the three-dimensional target space is treated as a sigma-model with the target space metric (B.3)

$$\mathrm{d}s^2_{\text{target}} = \sqrt{-\kappa}\left(-\frac{\mathrm{d}\kappa^2}{\kappa^2} + \frac{\mathrm{d}\omega^2 + \mathrm{d}\lambda^2}{\lambda^2}\right), \tag{2.5}$$

which is conformal to $\mathbb{R}\times H_2$ (where H_2 is Lobatchevski plane). The corresponding sigma model action (2.2) is

$$S = -\int_S \mathrm{d}^3x\sqrt{\hat{h}}\sqrt{-\kappa}\left(\frac{\hat{\mathcal{D}}^a\kappa\hat{\mathcal{D}}_a\kappa}{2\kappa^2} + 2\frac{\hat{\mathcal{D}}^a\tau\hat{\mathcal{D}}_a\tilde{\tau}}{(\tau-\tilde{\tau})^2} + \hat{\mathcal{R}} - 4i\Lambda\frac{\kappa}{\tau-\tilde{\tau}}\right). \tag{2.6}$$

Solving Einstein equations therefore is equivalent to studying the equations of motions of particles with on three dimensional spacetime with metric (2.5), interacting with a scalar potential with gravitational contributions $\sqrt{-\kappa}\hat{\mathcal{R}}$ (since the metric $\hat{h}$ is frozen),

$$V = \sqrt{-\kappa}\left(\hat{\mathcal{R}} - 2\Lambda\frac{\kappa}{\lambda}\right). \tag{2.7}$$

[2]See Appendix B for details

Let us now work with the simpler case when S is topologically $\mathbb{R}\times S_2$ and the metric $\hat{h}$ is chosen to have the form

$$\hat{h} = \mathrm{d}\Omega\otimes\mathrm{d}\Omega + \frac{\lambda^2}{\kappa^3}\mathrm{d}\hat{r}\otimes\mathrm{d}\hat{r}, \tag{2.8}$$

where $\mathrm{d}\Omega^2$ is necessarily a metric on S_2, E_2 or H_2 with $\hat{\mathcal{R}} = 2l$ for $l = 1, 0, -1$. We will later show that this choice of (2.3), allows us to use this mini-superspace ansatz to explore the entire space of solutions. The coordinate $\hat{r}$ here is chosen such that cosmological constant appears in the Hamiltonian as just a constant potential and can be treated as a constant of motion. Indeed, using the generalized momenta

$$p_\kappa = \frac{1}{\lambda}\frac{\mathrm{d}\kappa}{\mathrm{d}\hat{r}}, \qquad p_\omega = -\frac{\kappa^2}{\lambda^3}\frac{\mathrm{d}\omega}{\mathrm{d}\hat{r}}, \qquad p_\lambda = -\frac{\kappa^2}{\lambda^3}\frac{\mathrm{d}\lambda}{\mathrm{d}\hat{r}}, \tag{2.9}$$

the Hamiltonian is

$$\hat{H} = \frac{\lambda}{2}p_\kappa^2 - \frac{\lambda^3}{2\kappa^2}(p_\omega^2 + p_\lambda^2) + 2l\frac{\lambda}{\kappa} - 2\Lambda. \tag{2.10}$$

The Geroch algebra (Killing fields obeying $SL(2,\mathbb{R})$ group) is then realized in terms of the following Poisson brackets[3]

$$\begin{aligned}
\{\hat{H}, \hat{F}_+\} &= 0, \\
\{\hat{H}, \hat{F}_2\} &= -\hat{H} - 2\Lambda, \\
\{\hat{H}, \hat{F}_-\} &= 2\omega\hat{H} + 4\Lambda\left(\omega + \frac{\hat{r}\lambda^3 p_\omega}{\kappa^2}\right).
\end{aligned} \tag{2.11}$$

where $\hat{F}_+$, $\hat{F}_2$, and $\hat{F}_-$ are the following phase-space functions:

$$\begin{aligned}
\hat{F}_+ &= p_\omega, \\
\hat{F}_- &= -2\omega\lambda p_\lambda - (\omega^2 - \lambda^2)p_\omega - 4\Lambda\omega\hat{r}, \\
\hat{F}_2 &= \omega p_\omega + \lambda p_\lambda + 2\Lambda\hat{r}.
\end{aligned} \tag{2.12}$$

From (2.11) it is clear that for $\Lambda = 0$, all three dynamical functions $\hat{F}_\pm$ and $\hat{F}_2$ are conserved on the $\hat{H} = 0$ surface. However, when $\Lambda \neq 0$, only $\hat{F}_+$ and $\hat{F}_2$ are conserved. This corresponds to $N \subset SL(2,\mathbb{R})$ of section 2.1. Upon solving the Hamilton-Jacobi equations[4] for (2.10), one gets [8]

$$\begin{aligned}
\kappa &= -\frac{\Delta}{2\alpha}, \\
\lambda &= -\frac{\Delta}{2\alpha(r^2 + n^2)}, \\
\omega &= -\frac{n}{3\alpha}\left(\Lambda r + \frac{3lr - 3m - 4\Lambda n^2 r}{r^2 + n^2}\right),
\end{aligned} \tag{2.13}$$

[3]Poisson bracket is defined as $\{F, G\} = \sum_\mu \left(\frac{\partial F}{\partial p^\mu}\frac{\partial G}{\partial q^\mu} - \frac{\partial G}{\partial p^\mu}\frac{\partial F}{\partial q^\mu}\right)$ and $\frac{dF}{d\hat{r}} = \frac{\partial F}{\partial \hat{r}} + \{\hat{H}, F\}$

[4]The reader is reminded that Hamilton-Jacobi equations

$$\hat{H}\left(\frac{\partial S}{\partial q^i}, q^i\right) + \frac{\partial S}{\partial \hat{r}} = 0,$$

are just a canonical transformation involving a type-2 generating function $G(q, \alpha_i, \hat{r}) = S(q, \alpha_i, \hat{r}) + A$, where S is the principal solution and A is an arbitrary constant.

where α plays the role of a normalization constant and $n = p_\omega/2\sqrt{2\alpha}$. We have also changed coordinates to r such that

$$\hat{r} = \frac{1}{\sqrt{2\alpha}}\left(\frac{r^3}{3} + rn^2\right). \tag{2.14}$$

If we set

$$\alpha = \frac{m^2 + l^2 n^2}{2}, \tag{2.15}$$

then the 4 dimensional metric that is constructed from (2.13) following the procedure outlined in section 1.1 turns out to be the Taub-NUT AdS metric (A.4) with (m, n) as the mass and NUT charge and cosmological constant Λ.

We are now in a position where we can justify the need to explicitly consider the dynamics of the scalar field κ for AdS spacetime: For vanishing Λ of section 1.1, the (m, n) dependence of the metric λh ($= \kappa\hat{h}$ by equation (2.3)) could be absorbed into a single variable by a redefinition of the radial coordinate and κ only depended on this variable, i.e.,

$$\kappa = -\sinh^2\sigma = 1 - \left(\frac{r-m}{\sqrt{m^2+n^2}}\right)^2. \tag{2.16}$$

Hence κ is constant over the (m, n) parameter space under (1.11) and we could neglect its dynamics. However, for non-vanishing Λ, neither m nor n could be eliminated from κ and we had to consider the full dynamics in (2.4), which ultimately lead to a potential (2.7). This potential broke the symmetry group $SL(2,\mathbb{R})$ down to a subgroup henceforth called the $\mathcal{U}$-duality group, and only this subgroup generates algebraically new solutions *à la* Geroch.

2.3 $\mathcal{U}$-Duality

The sigma-model described by (2.6) is in general not integrable because the $SL(2,\mathbb{R})$ group is not large enough to account for the infinite number of conserved charges that are necessary for integrability [9, 17]. It however exhibits a continuous group of symmetries, the $\mathcal{U}$-duality group, which nonetheless allows one solution to be mapped to another.

However, because the solution generating group is only a subgroup of the $SL(2,\mathbb{R})$, its benefits for generating new solutions is limited. In particular, this group does not include the generator $m + in \to e^{-i\theta}(m + in)$, where θ is the angle of rotation in the parameter space (m, n) in the sense of (1.11). It is unfortunate because Schwarzschild-TaubNUT solutions exist on AdS and it is genuine to ask if there exists a map from Schwarzschild-AdS solution to TaubNUT-AdS solution similar to the vacuum case of section 1.3 (we do have similar transformations but all of them change Λ. Hence, for constant Λ, i.e., for AdS this is not possible).

In conclusion, for the $\Lambda \neq 0$ case, Geroch's algorithm is not sufficient to describe the complete solution generating technique and integrability for Einstein solutions. Hence, another approach is warranted. A possible candidate for this is provided by the gauge/gravity correspondence, which is going to be the topic for chapter 3.

Chapter 3

Holographic solution generation

The AdS/CFT correspondence was built based on the existence of Fefferman-Graham gauge, where the boundary data can be extended into the bulk. For a 4-dimensional bulk, the boundary is endowed with a boundary metric (containing 6 degrees of freedom when boundary is 3-dimensional) and a field theory symmetric-traceless stress-tensor (containing 5 degrees of freedom in 3-dimensions). These 5 degrees of freedom can be precisely recast as 5 degrees of fluid data which will be described in details in chapter 4 and appendix E, F. In this chapter, we will show results that support the claim that holography provides a new solution generating algorithm (these results will be "derived" in chapter 4). However, remember that the boundary data that we use in this solution generating algorithm (a boundary metric and stress tensor) are apriori not related to bulk data which are related by Geroch. Finding this relation is precicely one of the two questions that we proposed in the introduction (also see figure 4.1 and 4.2).

3.1 From 4D to 3D and back again

Starting from the bulk Einstein space $(\mathcal{M}, g)$, the holographic dictionary provides a boundary space $(\mathcal{B}, g_{\text{bdry}})$. The dictionary also states that fields in the bulk are related to operators on the boundary. Hence, for a purely gravitational system, the complete information of the bulk $(\mathcal{M}, g)$ is stored in the boundary data $(\mathcal{B}, g_{\text{bdry.}}, T)$.

Therefore to get a solution generating algorithm using holography, one would have to answer two questions: when is an exact bulk reconstruction possible using the boundary data and which allowed boundary transformations give new solutions? Authors in [9–14] have proposed a set of integrability requirements, which when obeyed by the boundary data, an exact bulk can be constructed by resummation of the derivative expansion of the boundary data. These integrability requirements also point out towards the possibility of new integrable solutions $(\mathcal{B}, g'_{\text{bdry}}, T')$ for any given boundary data $(\mathcal{B}, g_{\text{bdry}}, T)$. This is reminiscent of the dimensional reduction of Geroch's algorithm of section 1.1, and hence once could reconstruct new bulk solution by following scheme:

$$(\mathcal{M}, g) \underset{r\to\infty}{\longrightarrow} (\mathcal{B}, g_{\text{bdry.}}, T) \underset{\mathcal{U}_{\text{hol.}}}{\longrightarrow} (\mathcal{B}, g'_{\text{bdry}}, T') \underset{\text{exact reconstruction}}{\longrightarrow} (\mathcal{M}, g')$$

where r and $\mathcal{U}_{\text{hol.}}$ are the holographic radial coordinate and the $\mathcal{U}$-duality group respectively. In this chapter we will discuss these integrability conditions, the solution generating transformations and then show its application through the example of Plebański-Demiański Einstein spaces.

3.2 Holographic integrability

For pure gravity systems, the holographic dictionary suggests that in Fefferman-Graham gauge, the holographic radial coordinate is the field theory energy scale and hence holography is itself equivalent to a Hamiltonian evolution. The holographic integrability question then reduces to a *filling-in-problem*, i.e., when can an analytic three-dimensional metric be foliated by a four-dimensional Einstein space? Observations of [19] hint that the answer to this question is related to the conformal self-duality of Weyl tensor, i.e., there exists a certain relation between the boundary stress tensor and the Cotton tensor defined in (F.18). Similar observations from Fefferman-Graham expansion at order by order in Weyl tensor for large-r [20, 21], motivated the authors of [9–13] to look at the canonical reference energy-momentum tensors (associated with fictitious conserved boundary sources),

$$T^{\pm}_{\mu\nu} = T_{\mu\nu} \pm \frac{i\kappa}{3k^3} C_{\mu\nu}, \tag{3.1}$$

which are by definition symmetric, traceless and conserved, i.e.,

$$\nabla . T^{\pm} = 0. \tag{3.2}$$

Indeed if $T^{\pm}$ obeys the following conditions:

$$\begin{aligned} C &= \frac{3k^3}{\kappa} \mathrm{Im} T^{+}, \\ T &= \mathrm{Re} T^{+}, \end{aligned} \tag{3.3}$$

then a generic boundary metric, accompanied with canonical reference tensors $T^{\pm}$ of a perfect fluid form guarantees integrability (see the integrability conditions in 4.1 and the r^{-3} terms in (F.11)). Furthermore, the resummed bulk metric is Einstein. In chapter 4, we show the explicit form of this metric (4.6), starting from a boundary metric of the form (4.3). For sheerless boundary congruence, whenever the resummed metric is Einstein, it has to be algebraically special[1], i.e. of Petrov type II, III, D, N or O. The Petrov type of the bulk metric is dictated by the Segré type of the reference EM tensor, which can be either perfect-fluid, pure-radiation or pure-matter (or any combination). Hence, (3.2) and (3.3) can be thought of as an *holographic alternative of Geroch's three dimensional sigma model* (2.6).

Indeed (3.2) and (3.3) are invariant under a rotation in the (C, T)-plane, i.e.,

$$\mathrm{T}^{+} \longrightarrow \mathrm{T}'^{+} = \mathrm{z}\mathrm{T}^{+}, \qquad \mathrm{T}^{-} \longrightarrow \mathrm{T}'^{-} = \bar{\mathrm{z}}\mathrm{T}^{-}, \qquad \mathrm{z} \in \mathbb{C} \tag{3.4}$$

Hence any solution $(\mathcal{B}, g_{\mathrm{bdry.}}, T)$ of (3.2) and (3.3) can be mapped to another solution $(\mathcal{B}, g'_{\mathrm{bdry.}}, T')$, potentially new. This transformation although local at the level of actual stress tensor T (and Cotton tensor), is non-local for the boundary metric and the resummed bulk metric. Invariance of (3.2) under (3.4) can be recast as transformation of energy density and Cotton density as

$$\begin{pmatrix} \varepsilon \\ \frac{\kappa}{3k^3} c \end{pmatrix} \longrightarrow |z| \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} \varepsilon \\ \frac{\kappa}{3k^3} c \end{pmatrix} \tag{3.5}$$

with $z = |z|(\cos\psi + i \sin\psi)$. This transformation has the effect of rotation in (m, n) plane where the bulk "mass", m is associated with the real part of Ψ_2 component of Weyl tensor and the bulk NUT, n is associated with the imaginary part of Ψ_2 because for large r,

$$\Psi_2 \propto \varepsilon(x) + \frac{i\kappa}{3k^3} c(x). \tag{3.6}$$

This is precisely what (1.11) did for Ricci-flat spaces, which we failed to achieve for Einstein spaces in section 2.2. Notice that for $\psi = -\pi/2$ and $|z| = 1$, this is a gravitational duality map, which exchanges the mass and NUT charge: $(m, n) \to (-n, m)$.

[1]See Appendix A.2 of [10] for a proof of this statement.

We will now demonstrate the use of this holographic solution generating algorithm for Plebański-Demiański family of Einstein spaces with two-Killing symmetries in the next section. This will be interesting for two reasons: firstly, Plebański-Demiański spaces capture all aspects of black-hole physics and secondly, we will be able to retrieve the $\mathcal{U}$-duality group of section 2.2 and 2.3 through the non-trivial structure of the canonical reference energy-momentum tensors $T^{\pm}$.

3.3 Plebański-Demiański Einstein spaces

All known black hole solutions are Petrov type D [22] with two commuting Killing vectors, which itself is a subclass of Plebański-Demiański family [23], with an extra parameter associated with the black hole acceleration. We therefore expect that the boundary metric corresponding to the holographic dual of this metric has two isometries. To have a non-zero acceleration parameter, we need to ensure that $T^{\pm}$ are not proportional to each other, which forbids $\partial_t = \Omega u$ from being a timelike Killing vector. This warrants for a paramterization for the boundary metric different from the one used in (F.7) and (4.3). Inspired from [11], we chose the boundary metric to be

$$\mathrm{d}s^2 = -\frac{F-\chi^4 G}{F+G}\mathrm{d}\phi^2 + \frac{G-\chi^4 F}{F+G}\mathrm{d}\tau^2 + 2\chi^2\mathrm{d}\phi\mathrm{d}\tau + \frac{\mathrm{d}\chi^2}{FG}, \tag{3.7}$$

where τ and ϕ coordinates are adapted to the two Killing vectors and F and G are arbitrary functions of the third coordinate χ. Choosing the reference velocities as,

$$u_{\pm} = \frac{\partial_\tau \pm i\partial_\phi}{\chi^2 \mp i} \longleftrightarrow \mathrm{u} = \mathrm{d}\phi - \chi^2\mathrm{d}\tau + \frac{\mathrm{d}\chi}{F}, \tag{3.8}$$

For a perfect fluid, the reference tensor $T^{\pm}$ has the form of (E.2) where pressure is related to the Weyl one-form (F.10) by[2]

$$A^{\pm} + \frac{1}{3}\mathrm{d}\ln p_{\pm} = 0. \tag{3.9}$$

Using the above the form of congruence, together with (F.6) and (E.2) allows us to calculate the χ-dependent pressure $p_{\pm}(\chi)$ to be,

$$p_{\pm}(\chi) = -\frac{\kappa k}{3}\frac{m \mp n}{(\chi^2 \mp i)^3}, \tag{3.10}$$

with some arbitrary parameters m and n that will appear in the resummed bulk metric. Equations (3.2), (3.3) allow us to find the exact form of functions F and G up to some integration constants. With this we now have everything required to construct the resummed bulk metric which turns out to be the Plebański-Demiański metric [10].

We have therefore found an exact Einstein space starting from boundary data thanks to the integrability conditions (3.2) and (3.3). Therefore, starting from a given solution with parameters (m, n), another solution can be found through the application of (3.5). This is reminiscent of the $SL(2,\mathbb{R})/N$ subgroup of Geroch action, as was advertised.

In the next chapter, we will finally try to construct the holographic dual of Geroch group. To do this, we will use the derivative expansion used in fluid/gravity correspondence developed in [24–29] instead of the standard Fefferman-Graham expansion. We will do this because, firstly, derivative expansion can be resummed, leading to Einstein spacetimes in closed form, and secondly, it can be extended to asymptotically locally AdS [9–14] which allows a consistent vanishing scalar curvature, i.e., flatspace limit [14].

[2]The origin of these expressions is explained in detail in Chapter 4 and appendix E, F

Chapter 4

Fluid/gravity duality

In chapter 3, we were mostly concerned with developing a new solution generating algorithm that might contain the missing sector of chapter 2. We saw that holography is able to probe this missing sector and hence deserves to be studied in detail. This will be the goal of this chapter. An important tool for this will be the fluid/gravity dictionary and derivative expansion, which are outlined in section 4.1. In section 4.2 we study their Carrollian limit that emerges when scalar curvature vanishes. Finally, in section 4.3 we show the recent progress towards Geroch dual and some of the answers we have found. Appendix H provides an example of holographic reconstruction, where we reconstruct the Ricci-flat Kerr-Taub-NUT family of spacetime using the boundary Carrollian fluids.

4.1 Fluid/Gravity duality in AdS

The AdS/CFT correspondence tells us that the dynamics of the normalizable fall-off of the $d+1$-dimensional AdS metric at the boundary (i.e., boundary stress tensor) is that of a d-dimensional QFT. At high temperatures (large wavelength fluctuations), every interacting QFT equilibrates into a "fluid phase", i.e., a translationally invariant phase in which adiabatic displacement of neighbouring elements requires no force [24]. The fluid/gravity correspondence has been described in details in Appendix E and F. It can then be intuitively summarized as:

Fluid/gravity correspondence

A collection of large number of excitations of the QFT, that continually collide with a mean free time t_m (and mean free length l_m) can be characterized by density function, with their time evolution being governed by transport equations of statistical physics (for eg: Boltzmann equation). For sufficiently long wavelength fluctuations ($\lambda \gg l_m$), variations of local energy density/temperature are slow and hence locally at any point in the system we encounter a region of constant temperature. To this domain, we assign a grand canonical ensemble to extract the relevant thermodynamic quantities. Fluid dynamics then describes how these domains interact and exchange thermodynamic quantities.

This becomes possible because we can describe the 5 degrees of freedom contained in the symmetric-traceless boundary stress tensor (as discussed in the introduction of chapter 3) with 5 degrees of freedom of the boundary fluid (one in energy density, ϵ, two in heat current, q^μ and two in symmetric-traceless viscous stress tensor, $\tau_{\mu\nu}$ of equation (4.1), provided these are related to some derivatives of fluid velocity and temperature). With this equivalence in mind, we summarize how to construct asymptotically locally AdS-Einstein spacetimes in Eddington-Finkelstein gauge (along null congruence) starting from boundary fluid data.

The two pieces of boundary data that we require are the *fluid stress tensor* and the *boundary metric*:

Boundary stress tensor Consider a null congruence of geodesics that are timelike on the boundary and extend into the bulk. The timelike boundary congruence is identified with a boundary fluid which is characterized by normalized fluid velocity u (with normalization $||u||^2 = -k^2$), acceleration a, expansion Θ, shear $\sigma_{\mu\nu}$ and vorticity $\omega_{\mu\nu}$, given in (E.9). Then the boundary stress tensor can be written in the form (E.10), i.e.,

$$T_{\mu\nu} = (\epsilon + p)\frac{u_\mu u_\nu}{k^2} + p g_{\mu\nu} + \tau_{\mu\nu} + \frac{u_\mu q_\nu}{k^2} + \frac{u_\nu q_\mu}{k^2}, \tag{4.1}$$

where the fluid energy density ϵ and pressure p are related by conformal equation of state $\epsilon = 2p$. The boundary energy-momentum tensor is conserved and its dynamics is governed by the hydrodynamic equation (E.4). The Cotton tensor defined in (F.18), which we stressed in chapter 3 as an 'alternative' description of the boundary metric, vanishes when the bulk is asymptotically globally AdS (and the boundary is conformally flat). For locally asymptotically AdS it can be extended just like (4.1),

$$C_{\mu\nu} = \frac{3c}{2}\frac{u_\mu u_\nu}{k} + \frac{ck}{2} g_{\mu\nu} - \frac{c_{\mu\nu}}{k} + \frac{u_\mu c_\nu}{k} + \frac{u_\nu c_\mu}{k}, \tag{4.2}$$

where $c_{\mu\nu}$ is the Cotton stress tensor, c_μ is the Cotton current and c the Cotton scalar density given explicitly in (F.22), (F.23) and (F.24) respectively. Notice the similarity between the forms of energy-momentum tensor and the Cotton tensor in (4.1) and (4.2). This similarity is the basis of our claims of chapter 3 and equation (3.3) in particular.

Boundary metric The boundary metric is chosen to be parametrized *à la* Randers-Papapetrou as in (F.3). The velocity one-form and the metric then are,

$$\begin{aligned} \underline{u} &= -k^2(\Omega \mathrm{d}t - b_i \mathrm{d}x^i), \\ \mathrm{d}s^2_{bry} &= -k^2(\Omega \mathrm{d}t - b_i \mathrm{d}x^i)^2 + a_{ij}\mathrm{d}x^i \mathrm{d}x^j. \end{aligned} \tag{4.3}$$

In this frame, temporial components of heat current and viscous stress tensor vanish, i.e., $q_0 = \tau_{00} = \tau_{i0} = \tau_{0i} = 0$. Similarly, $c_0 = c_{00} = c_{i0} = c_{0i} = 0$.

Bulk resummation Building upon the observation of [24–29] that the bulk geometry should be insensitive to a conformal rescalings of boundary metric (since it corresponds to bulk diffeomorphisms), authors of [9–14] were able to reconstruct the four dimensional bulk metric order by order in derivatives of u. Under conformal rescalings, $\mathrm{d}s^2_{\text{bdry}} \to \mathrm{d}s^2_{\text{bdry}}/\mathcal{B}^2$, and hence, the boundary metric has a weight -2. Similarly $r, T_{\mu\nu}$, $\tau_{\mu\nu}$ have weight 1; ϵ, p have weight 3; q_μ has weight 2; $\underline{u}$ has a weight -1 and $\omega_{\mu\nu}$ (vorticity two-form) has weight -1. This motivates us to define a Weyl connection one form (F.10),

$$\mathrm{A} = \frac{1}{k^2}\left(\mathrm{a} - \frac{\Theta}{2}u\right), \tag{4.4}$$

where the explicit form of the fluid parameters a and Θ in the boundary metric (4.3) are given in (F.6). We also define the corresponding Weyl covariant geometric quantities like the Weyl covariant derivative $\mathscr{D}_\rho$ (F.9), Riemman tensor $\mathscr{R}_{\mu\nu\rho}{}^{\sigma}$ (F.20), Ricci tensor $\mathscr{R}_{\mu\nu}$ (F.21) etc. (see appendix F.)

Given all ingredients listed above, the expansion of the bulk Einstein metric was shown to be of the form (F.11). Further substituting,

$$\begin{aligned} r^2/\rho^2 &\to 2 - \rho^2/r^2, \quad \text{where} \\ \rho^2 &= r^2 + \gamma^2. \end{aligned} \tag{4.5}$$

and γ is the scalar vorticity given by (F.17), allows us to resum the bulk metric (F.11) in a closed form, provided the fluid is shearless, to give

$$\mathrm{d}s^2_{\text{res. Einstein}} = 2\frac{\mathrm{u}}{k^2}(\mathrm{d}r + r\mathrm{A}) + r^2 \mathrm{d}s^2 + \frac{\mathrm{S}}{k^4} + \frac{\mathrm{u}^2}{k^4\rho^2}\left(8\pi G\varepsilon r + c\gamma\right), \tag{4.6}$$

where S is a Weyl-invariant tensor, given by (F.12).

Shearlessness This assumption of shearless congruence considerably reduces the number of terms appearing in (F.11) and hence deserves to be discussed in details. First, notice that this freedom to choose a shearless congruence originates from the fact that for relativistic fluids, it is impossible to distinguish heat current from matter flux, leaving one degree of freedom in choosing the velocity field (see appendix E). Assuming a shearless congruence is equivalent to demanding the two-dimensional spatial section $\mathscr{S}$ defined at every time and equipped with metric $\mathrm{d}l^2 = h_{ij}\mathrm{d}x^i\mathrm{d}x^j$ which appears in boundary metric (F.7) is conformally flat[1]. In $d = 3$ it is always possible to find such a shearless congruence from which, one can recover all Petrov-algebraically special AdS spacetimes [12].

Interestingly, in flat spacetimes this assumption is a genuine fact, because the boundary sheer is proportional to bulk sheer and cosmological constant as a consequence of Einstein's equations, and hence boundary shear has to vanish for $\Lambda = 0$. Furthermore, equation (F.6), tells us that whenever there is a timelike Killing field (which we will assume in section 4.3), boundary shear vanishes naturally.

Holographic integrability The metric (4.6) is an exact Einstein space, with cosmological constant, $\Lambda = -3k^2$ iff the following conditions are satisfied:

- *The congruence u is shearless.*
- *The heat current of the boundary fluid introduced in* (4.1) *is identified with the transverse-dual of the Cotton current defined in* (4.2) *and* (F.23)., i.e.,

$$q_\mu = \frac{1}{8\pi G}\eta^\nu{}_\mu c_\nu, \tag{4.7}$$

  where $\eta^\nu{}_\mu$ is defined in (F.15). Using holomorphic and antiholomorphic coordinates $\zeta, \bar\zeta$ as in (F.7)[2] leads to $\eta^\zeta{}_\zeta = \mathrm{i}$ and $\eta^{\bar\zeta}{}_{\bar\zeta} = -\mathrm{i}$, and thus

$$\mathrm{q} = \frac{\mathrm{i}}{8\pi G}\left(c_\zeta \mathrm{d}\zeta - c_{\bar\zeta}\mathrm{d}\bar\zeta\right). \tag{4.8}$$

- *The viscous stress tensor of the boundary conformal fluid introduced in* (4.1) *is identified with the transverse-dual of the Cotton stress tensor defined in* (4.2) *and* (F.24). Following the same pattern as for the heat current, we obtain:

$$\tau_{\mu\nu} = -\frac{1}{8\pi G k^2}\eta^\rho{}_\mu c_{\rho\nu}, \tag{4.9}$$

  and in complex coordinates:

$$\tau = -\frac{\mathrm{i}}{8\pi G k^2}\left(c_{\zeta\zeta}\mathrm{d}\zeta^2 - c_{\bar\zeta\bar\zeta}\mathrm{d}\bar\zeta^2\right). \tag{4.10}$$

- *The energy-momentum tensor must be conserved, i.e., it must obey* (E.4).

The conditions stated above are exactly the same as (3.1), (3.2), and (3.3) that were discussed with regards to the holographic solution generating algorithm in chapter 3. The cannonical stress tensor of (3.1), $T^\pm$ in complex coordinates are given by

[1]Notice that any two-dimensional metric can always be brought into a conformally flat form, however, here we wish to make $\mathrm{d}l^2$ conformally flat while ensuring that the three-dimensional metric is given by (4.3). This is possible only if the transformations used to make it flat are of the form $x' = x'(x)$, i.e., if the transformations have no t dependence

[2]Orientation is chosen such that in the coordinate frame $\eta_{0\zeta\bar\zeta} = \sqrt{-g}\epsilon_{0\zeta\bar\zeta} = \frac{\mathrm{i}\Omega}{P^2}$, where $x^0 = kt$.

$$\mathrm{T}^{+} = \varepsilon_{\pm}\left(\frac{\mathrm{u}^2}{k^2} + \frac{1}{2}\mathrm{d}\ell^2\right) + \frac{\mathrm{i}}{4\pi G k^2}\left(2c_{\zeta}\mathrm{d}\zeta\mathrm{u} - c_{\zeta\zeta}\mathrm{d}\zeta^2\right), \tag{4.11}$$

With,

$$\varepsilon_{\pm} = \varepsilon \pm \frac{\mathrm{i}c}{8\pi G}. \tag{4.12}$$

This should have been expected because the holographic solution generating algorithm was also based on holographic integrability. Identifying parts of the energy–momentum tensor with the Cotton tensor may be compared to the electric–magnetic duality conditions in electromagnetism with (E.4) as a generalization of Gauss law. Note that the only independent boundary data now is a_{ij}, b_i, Ω and ϵ.

With this, we are now equipped to tackle one of the two questions that were raised at the start of this report: *"How is the holographic solution generating algorithm of chapter 3 related to the holographic duals of* $(\mathcal{M}, g)$ *and* $(\mathcal{M}, g')$*, where* g *and* g' *are related à la Geroch*?" This question can be visualized through figure 4.1:

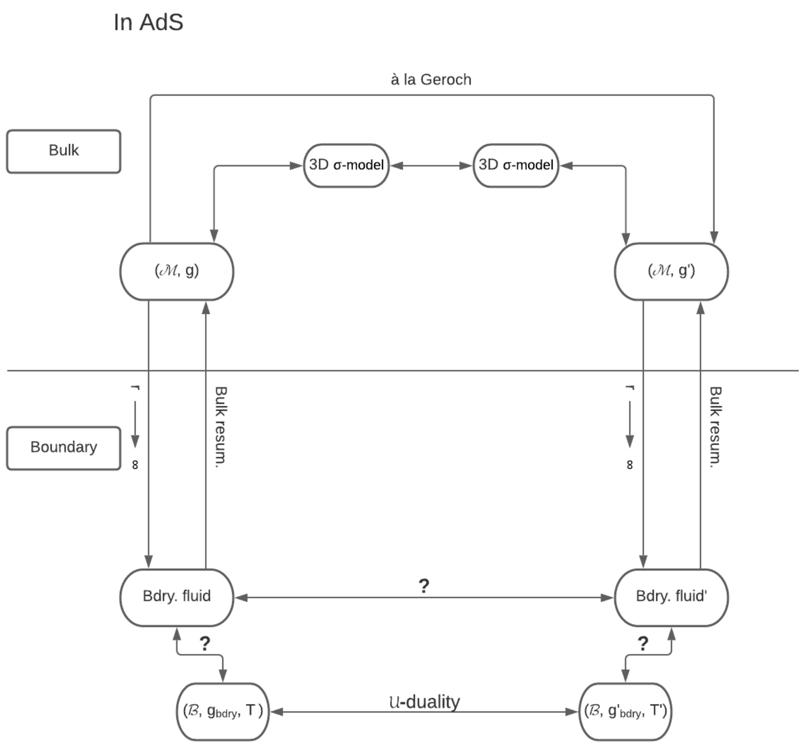

Figure 4.1: In AdS case, we are interested in understanding the 2 sets of question-marks: how are the boundary fluids (whose bulk duals are related by Geroch) related? and how are these boundary fluids related to the solution generating algorithm, $\mathcal{U}$-duality group of chapter 3?

To answer this, one needs to start with the bulk metric of (4.6), assuming a Killing vector (say ∂_t) and apply Geroch's algorithm of section 1.1, to obtain a new 'resummed metric'. We expect that both the original and the new bulk metrices will correspond to different boundary data, which however obey the same fluid dynamical equations, (E.4). This will be the equivalent of sigma-model equations (2.2). However, for now, we will postpone these calculations to section 4.3 and focus on developing flatspace holography in section 4.2, so that we can tackle this and "Can holography provide new insights for the Ricci-flat case" together.

4.2 Carrollian limit: Flat holography

As stated in the last section, the metric (4.6) is an exact Einstein space, with cosmological constant $\Lambda - 3k^2$. This k was introduced in our analysis as through the normalization of boundary fluid, i.e., $||u||^2 = -k^2$. Hence, the Ricci-flat limit $k \to 0$, corresponds to fluid velocity being a null congruence. This limit is also referred to as the "Carrollian limit" [30–33]. Derivative expansion is well equipped to handle this limit because derivative expansion constructs Einstein spacetimes in Eddington-Finkelstein gauge, i.e., the expansion is performed along a null congruence. In this section we will first study what this limit physically means, and then its consequences for the fluid/gravity correspondence.

4.2.1 Carrollian algebra

For vanishing k[3], time decouples in the Randers–Papapetrou metric (4.3). There are two limits in which time decouples from rest of the metric, associated with two distinct contractions of the Poincaré group: the Galilean, reached at infinite velocity of light and referred to as "non-relativistic", and the Carrollian, emerging at zero velocity of light, often called "ultra-relativistic". Indeed both Gallelian, $t' = t'(t)$ & $\mathbf{x}' = \mathbf{x}'(t, \mathbf{x})$ and Carrollian, $t' = t'(t, \mathbf{x})$ & $\mathbf{x}' = \mathbf{x}'(\mathbf{x})$ transformations are a subset of the diffeomorphism group. For reference, let us briefly review the Gallelian group and infinite-c contraction.

Consider a free particle in an arbitrary d-dim. space $\mathscr{S}$, endowed with metric,

$$\mathrm{d}\ell^2 = a_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{4.13}$$

We can define an action which is invariant under Gallelian diffeomorphisms $t' = t'(t)$ & $\mathbf{x}' = \mathbf{x}'(t, \mathbf{x})$ (and hence, has a Newton-Cartan structure on it) as,

$$S[\mathbf{x}] = \int_{\mathscr{C}} \mathrm{d}t\, \Omega \mathcal{L}(\mathbf{v}, \mathbf{x}, t), \quad \text{with} \tag{4.14}$$

$$\mathcal{L}(\mathbf{v}, \mathbf{x}, t) = \frac{1}{2\Omega^2} a_{ij}\left(v^i - w^i\right)\left(v^j - w^j\right). \tag{4.15}$$

where w^i and v^i represent the velocity of a local inertial frame, and the velocity of a particle in it respectively. One can check that this Lagrangian indeed gives the standard $\mathcal{H} = \vec{p}^2/2$, free particle Hamiltonian for $\vec{w} = 0$. In order to see that the Galilean group is the infinite-c contraction of the Poincaré group, consider a $d+1$-dimensional pseudo-Riemannian manifold $\mathscr{M}$, endowed with a metric[4],

$$\mathrm{d}s^2 = -\Omega^2 c^2 \mathrm{d}t^2 + a_{ij}\left(\mathrm{d}x^i - w^i\mathrm{d}t\right)\left(\mathrm{d}x^j - w^j\mathrm{d}t\right). \tag{4.16}$$

The Galilean limit requires Ω to depend only on t[5]. Dynamics of a relativistic particle on (4.16) can then be described using the length of the world-line $\mathscr{C}$, using the action parametrized by proper time for a physical observer, $\tau = \sqrt{-\mathrm{d}s^2/c^2}$ as,

$$S[x] = \int_{\mathscr{C}} \mathrm{d}\tau = \int_{\mathscr{C}} \sqrt{-\frac{\mathrm{d}s^2}{c^2}}. \tag{4.17}$$

Expanding for large-c,

$$S[x] = \int_{\mathscr{C}} \mathrm{d}t\, \Omega \left(1 - \frac{1}{2c^2\Omega^2} a_{ij}\left(v^i - w^i\right)\left(v^j - w^j\right) + \mathrm{O}\left(1/c^4\right)\right), \tag{4.18}$$

[3]Since k, the speed of light vanishes in the Carrollian limit, motion is forbidden. This might sound alarming as one would think that no dynamics can emerge in such a system. However, extended instantonic objects do still exist and have non-trivial dynamics, making this framework rich and interesting. In fact, Carrollian dynamics turns out be even richer than its Gallelian counterpart (see section 4.2.2)

[4](4.16) referred to as Zermelo metric, is the analog of Randers-Papapetrou metric (4.3)

[5]This is because the Newtonian absolute time, $\int \Omega \mathrm{d}t$ must coincide with the proper time τ.

which indeed matches with (4.14) and (4.15) by ignoring the constant (and hence, Galilean invariant) term. This shows that (4.16) is the natural relativistic spacetime uplift of a Galilean space $\mathscr{S}$ endowed with a Newton–Cartan structure.

In an analogous fashion, one can show that the Randers-Papapetrou metric (4.3) is the natural "ultra-relativistic" uplift of Carrollian space $\mathscr{S}$ endowed with a Carrollian structure (i.e., a Carrollian fluid in a frame with 'inverse velocity'[6] $\boldsymbol{b} = b_i(t, \boldsymbol{x})\mathrm{d}x^i$ and a scale density factor $\Omega(t, \boldsymbol{x})$) [15]. See appendix G where we work this out explicitly.

4.2.2 Carrollian 'hydrodynamics' and flat holography

In AdS case, the conformal boundary was timelike, however, for the flat case, the boundary congruence is lightlike. As discussed in section 4.2.1, this corresponds to Carrollian limit, where time decouples from the boundary metric. Hence, the Carrollian structure resides on a spacetime $\mathbb{R} \times \mathscr{S}$[7], where $t \in \mathbb{R}$ is the Carrollian time and the spatial surface $\mathscr{S}$ which appears at the null infinity of the Ricci-flat geometry is endowed with a positive-definite metric (4.13). The boundary energy-momentum tensor and boundary metric can be parametrized using (4.1) and (4.3) just like in section 4.2.1. The Carrollian energy and pressure are just the zero-k limits of the corresponding relativistic quantities and obey the same equation of state. For non-ideal components, the $k \to 0$ limit requires us to consider

$$Q_i = \lim_{k\to 0} q_i, \quad \pi_i = \lim_{k\to 0} \tfrac{1}{k^2}\left(q_i - Q_i\right), \tag{4.19}$$

$$\Sigma_{ij} = -\lim_{k\to 0} k^2 \tau_{ij}, \quad \Xi_{ij} = -\lim_{k\to 0}\left(\tau_{ij} + \tfrac{1}{k^2}\Sigma_{ij}\right). \tag{4.20}$$

where $\boldsymbol{Q}$, $\boldsymbol{\pi}$ are the heat currents and $\boldsymbol{\Sigma}$, $\boldsymbol{\Xi}$ are the viscous stress tensors. Note that the degrees of freedom are doubled compared to their Galilean counterparts, as promised in footnote 3. Similarly, the zero-k limit of Cotton gives

$$\chi_i = \lim_{k\to 0} c_i, \quad \psi_i = \lim_{k\to 0} \tfrac{1}{k^2}\left(c_i - \chi_i\right), \tag{4.21}$$

$$X_{ij} = \lim_{k\to 0} c_{ij}, \quad \Psi_{ij} = \lim_{k\to 0} \tfrac{1}{k^2}\left(c_{ij} - X_{ij}\right). \tag{4.22}$$

Ricci-flat bulk Resummation Similar to the AdS case, we use Weyl invariance of boundary geometry to get the resummed metric. Analogous to (4.4), we define a Weyl-Carroll covariant derivative $\hat{\mathscr{D}}_i$ (G.21), Riemman-Weyl-Carroll tensor $\hat{\mathscr{R}}_{\mu\nu\rho}{}^{\sigma}$ (G.24), Ricci-Weyl-Carroll tensor $\hat{\mathscr{R}}_{\mu\nu}$ (G.25) etc, where Carrollian quantities are represented with a hat (due to the redefinition of partial derivatives (G.10)). The Carrollian fluid data, i.e., Carrollian fluid acceleration φ_i, expansion θ, sheer ξ_{ij} and vorticity ϖ_{ij} can also be defined in a similar fashion by replacing the derivatives in (F.6) by their Carrollian counterparts,

$$\begin{aligned} \varphi_i &= \partial_t \frac{b_i}{\Omega} + \hat{\partial}_i \ln \Omega, \\ \theta &= \frac{1}{\Omega}\partial_t \ln\sqrt{a}, \end{aligned} \tag{4.23}$$

[6] In Carrollian spacetimes motion is forbidden so we define a temporial frame using $\boldsymbol{b}$

[7] In the previous section, we showed that the Carrollian structure is hosted by $\mathscr{S}$ and hence this can effectively be thought to replace AdS boundary for flat case. The idea of a Ricci-flat limit being related to some contraction of Poincaré group at null infinity is not new [34–38], however, these studies are accustomed to the standard CFT dynamics and relativistic fluids instead of Carrollian one, due to potential confusion caused by equivalence of Gallelian and Carrollian algebra in two dimensions. Hence, one should not forget that the boundary is still 3D, albeit the timelike direction is degenerate.

$$\xi_{ij} = \frac{1}{2\Omega} a^{ik}(\partial_t a_{kj} - a_{kj}\partial_t \ln\sqrt{a}),$$
$$\varpi_{ij} = \frac{\Omega}{2}\left(\hat{\partial}_i \frac{b_j}{\Omega} - \hat{\partial}_j \frac{b_i}{\Omega}\right). \tag{4.24}$$

Combining acceleration and expansion, we define an analog of Weyl connection (4.4),

$$\alpha_i = \varphi_i - \frac{\theta}{2} b_i, \tag{4.25}$$

which transforms appropriately under Weyl rescaling (G.18). Then the hydrodynamic equations (E.4) give,

$$-\frac{1}{\Omega}\hat{\mathscr{D}}_t \epsilon - \hat{\mathscr{D}}_i Q^i + \Xi^{ij}\xi_{ij} = 0,$$
$$\hat{\mathscr{D}}_j p + 2Q^i \omega_{ij} + \frac{1}{\Omega}\mathscr{D}_t \pi_j - \hat{\mathscr{D}}_i \Xi^i j + \pi_i \xi^i j = 0,$$
$$\frac{1}{\Omega}\hat{\mathscr{D}}_t Q_j - \hat{\mathscr{D}}_i \Sigma^i_j + Q_i \xi^i j = 0,$$
$$\Sigma^{ij}\xi_{ij} = 0, \tag{4.26}$$

where since the d.o.f. are doubled, the first two equations correspond to energy-momentum conservation (one scalar and one vector) and the latter set a geometrical constraint on Carrollian viscous stress tensor Σ_{ij} and heat current Q_i.

Using this data, the boundary metric can be resummed to give

$$\mathrm{d}s^2_{res.flat} = -2(\Omega \mathrm{d}t - \boldsymbol{b})\left(\mathrm{d}t + r\boldsymbol{\alpha} + \frac{r\theta\Omega}{2}\mathrm{d}t\right) + r^2 a_{ij}\mathrm{d}x^i \mathrm{d}x^j + \boldsymbol{s} + \frac{(\Omega \mathrm{d}t - \boldsymbol{b})^2}{\rho^2}(8\pi G \epsilon r + c \star \omega), \tag{4.27}$$

where

$$\rho = r^2 + \star\omega^2, \tag{4.28}$$

is the analog of (4.5) and $\boldsymbol{s}$ is a Weyl invariant tensor analogous to S for AdS, (G.27).

Holographic integrability Metric (4.27) is closed iff:

- *The heat current of the boundary Carrollian fluid introduced in* (4.19) *is identified with the transverse-dual of the Cotton current defined in* (4.21), i.e.,

$$Q_i = \frac{1}{8\pi G}\eta^j{}_i \chi_j,$$
$$\pi_i = \frac{1}{8\pi G}\eta^j{}_i \psi_j, \tag{4.29}$$

  where $\eta^\nu{}_\mu$ is defined in (F.15).

- *The viscous stress tensor of the boundary Carrollian fluid introduced in* (4.20) *is identified with the transverse-dual of the Cotton stress tensor defined in* (4.22). Following the same pattern as for the heat current, we obtain:

$$\Sigma_{ij} = \frac{1}{8\pi G}\eta^l{}_i X_{lj},$$
$$\Xi_{ij} = \frac{1}{8\pi G}\eta^l{}_i \Psi_{lj}, \tag{4.30}$$

- *The hydrodynamics equations* (4.26) are satifyed,

$$\begin{aligned} -\frac{1}{\Omega}\hat{\mathscr{D}}_t\epsilon + \frac{1}{16\pi G}\hat{\mathscr{D}}^i(\hat{\mathscr{D}}_i\hat{\mathscr{K}} - \eta^j{}_i\hat{\mathscr{D}}_j\hat{\mathscr{A}} + 4\star\omega\eta^j{}_i\hat{\mathscr{R}}_j) = 0, \\ \hat{\mathscr{D}}_j\epsilon + 4\star\omega\eta^j{}_iQ_j + \frac{2}{\Omega}\hat{\mathscr{D}}_t\pi_j - 2\hat{\mathscr{D}}_i\Xi^i_j = 0. \end{aligned} \tag{4.31}$$

  where the first two equations of (4.26) are satisfied naturally.

These conditions are analogous to those of section 4.1 and chapter 3. With this, we can in principle find the boundary data corresponding to a Ricci-flat bulk metric. An example of this is shown in Appendix H, where we explicitly construct Schwarzschild-TaubNUT solutions starting from boundary Carrollian fluid data. Now we can finally ask the question: *"Can holography provide new insights into Geroch algorithm for the Ricci-flat case?"* To answer this, we would like to apply Geroch's algorithm to (4.27), assuming it has a timelike Killing vector. This process is summarized using figure 4.2:

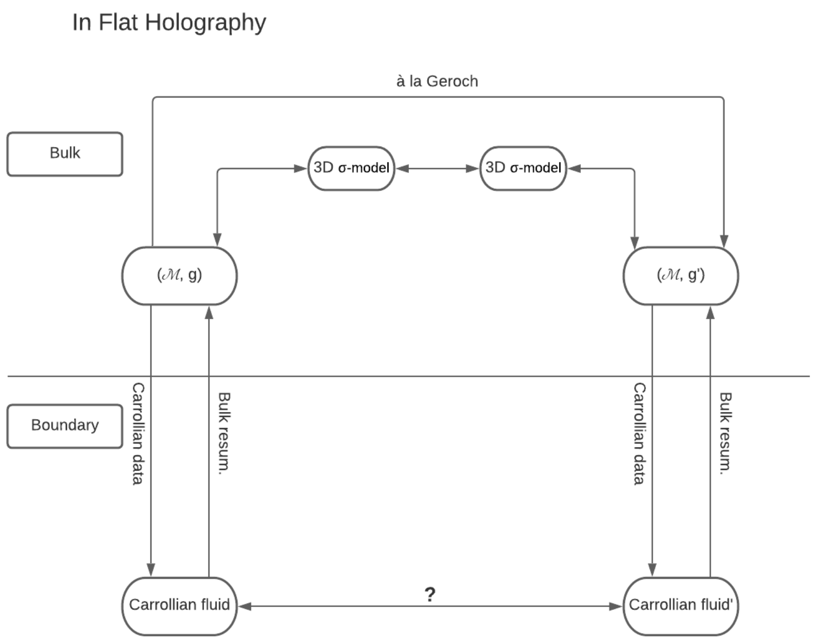

Figure 4.2: For the flat case, we would like to know how the boundary Carrollian fluids (whose bulk duals are related *á la* Geroch) transform into one another.

4.3 Geroch dual

In section 4.1 and 4.2 we have outlined all the key components required to construct the holographic dual. In this section I will briefly mention some of our results (preliminary results are reproduced in appendix G). First, let us make some physical remarks:

In AdS Apart from answering the question that was already motivated in the beginning, we also discovered in section 4.1 that the resummed metric (4.6) consists of all Petrov-algebraically special four-dimensional locally AdS-spacetimes. Applying a Geroch-like algorithm can then tell us "*Is the set of algebraically special Einstein spaces closed*", i.e., if we apply Geroch's algorithm, are we guaranteed that the new metric will be algebraically special? If yes, can we go from one Petrov type to the other? If not, can the new solutions still be parameterized using (4.6)?

In Flatspace A similar question regarding the stability of algebraically special flat-spacteimes can be answered via Geroch (probably more robustly owing to the incompleteness of Geroch in AdS). But more importantly, in section 1.3 we mentioned that ∂_ϕ does not generate new solutions starting from Schwarzschild metric. In fact, it is known that only timelike Killing vector generates new solutions for the equation of state $\epsilon = 2p$

and only spacelike Killing vector generates new solutions when $\epsilon = -p$ in three-dimensions [39]. Since for (4.27) we are using $\epsilon = 2p$, we expect only timelike Killings to generate new solutions. This can be physically motivated from holography in our formalism: the boundary corresponding to (4.27) is $\mathbb{R} \times \mathscr{S}$, where time like direction is degenerate (null), i.e., $t \in \mathbb{R}$. It is known from the literature on Newton-Cartan and Carroll that a reduction along null Killing leads to Newton-Cartan structure, while if a null hypersurface is embedded in a manifold, it has Carrollian structure [40]. Since bulk timelike Killing corresponds to null Killing on the boundary, Geroch's dimensional reduction along $\xi = \partial_t$ is like embedding a Newton-Cartan structure on top of Carrollian structure. Bulk spacelike Killings cannot to do this and hence cannot generate new solutions. It would be interesting to see if we can extend our formalism to $\epsilon = -p$ and see if we can motivate that only bulk spacelike Killings should lead to new solutions *á la* Geroch.

Calculationally, the first thing to do is to expand (4.27), into its components and assume that we have a timelike Killing field $\xi = \partial_t$, for which $\theta = 0$, which simplifies the Killing one form (I.1) and expanded metric (H.4) for the flat case to,

$$\underline{\xi} = \Omega\left(A\Omega \mathrm{d}t - \mathrm{d}r - (AL+B)\mathrm{d}Z - i(AD+Q)\mathrm{d}\Phi\right), \qquad \text{and} \tag{4.32}$$

$$\begin{aligned} \mathrm{d}s^2_{\text{res. flat}} = {} & \Omega^2 A\mathrm{d}t^2 - 2\Omega\mathrm{d}r\mathrm{d}t + \mathrm{d}Z^2\left(\frac{2\rho^2}{P^2} + L(AL+2B)\right) - 2\Omega\mathrm{d}t\mathrm{d}Z\,(AL+B) \\ & + 2L\mathrm{d}r\mathrm{d}Z + \mathrm{d}\Phi^2\left(\frac{2\rho^2 Z^2}{P^2} - D(AD-2Q)\right) + 2iD\mathrm{d}r\mathrm{d}\Phi \\ & - 2i\Omega\mathrm{d}t\mathrm{d}\Phi\,(AD+Q) + 2i\mathrm{d}Z\mathrm{d}\Phi(ALD+LQ+DB), \end{aligned} \tag{4.33}$$

where we have used the (t, r, Z, Φ)-coordinates such that $\zeta = Ze^{i\Phi}$ and the quantities appearing here, A, B and Q are functions of (t, r, Z, Φ), while L and D are functions of (t, Z, Φ), (H.3). Norm and vorticity of the Killing field (1.1) required for Geroch are

$$\begin{aligned} \lambda &= A\Omega^2, \\ \omega_a &= -g_{ae}\frac{\epsilon^{bcde}}{\sqrt{-g}}\left(\underline{\xi} \wedge \mathrm{d}\underline{\xi}\right)_{bcd}. \end{aligned} \tag{4.34}$$

We have explicitly worked them out for the metric (4.33) in Appendix I and verified that they match for Schwarzschild and Taub-NUT cases.

Conclusions and Outlook

In chapter 1, 2 of this report, we presented Geroch's solution generating algorithm and its extensions and demonstrated their importance through various examples. In chapter 3 and 4, we motivated how holography is able to probe these and provide interesting insights into both the solution generating algorithm and the solution space itself. For this, we relied on [9–16] where the fluid/gravity literature was expanded to show that one can recover all algebraically special bulk Einstein spaces using derivative expansion under certain integrability conditions. Fluid/gravity correspondence was also extended to flat spacetimes, and was shown to correspond to the Carrollian limit of its AdS counterpart.

Through this report, a formalism to answer many interesting questions was developed (see section 4.3 and figure 4.1, 4.2 for some of these questions and insights.) We have preliminary results that answers some of these, however, these are yet to be verified and hence have been omitted from the report.

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Appendix A

Taub-NUT geometries

The Taub-NUT solution is given by [23]

$$g = -f(r)\left(\mathrm{d}t - 2l\cos\theta\mathrm{d}\phi\right)^2 + \frac{\mathrm{d}r^2}{f(r)} + \left(r^2 + l^2\right)\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right), \tag{A.1}$$

where

$$f(r) = \frac{r^2 - 2mr - l^2}{r^2 + l^2}, \tag{A.2}$$

with constant parameters m and l, where l is known as the NUT parameter. The twist for each congruence is proportional to the NUT parameter, which is why l is also regarded as the twist parameter. In the limit when $l \to 0$, the metric reduces to Schwarzschild solution with mass m. It is always possible to make $m \geq 0$, by taking $r \to -r$. This is why this solution can be considered as an exterior field of rotating source with radius greater than r_+[1], where where $r_\pm$ are the roots of $f(r)$,

$$r_\pm = m \pm \sqrt{m^2 + l^2}. \tag{A.3}$$

Taub-NUT spacetime has many unexpected global properties. In the NUT region $r < r_-$ and $r > r_+$, r is spacelike, the metric is stationary and contains closed timelike curves (curves on which t, r and θ are constants). In the Taub region $r_- < r < r_+$, r is timelike and the metric is spatially-homogeneous[23]. All trajectories that enter Taub region, remain within it for the same finite amount of proper time, which is why this region was used to study cosmological models which are spatially-homogeneous, and more generally those which are hypersurface-homogeneous and self-similar [41]. The two solutions can be considered as part of a single manifold, being joined across a null hypersurface [1].

In terms of the natural tetrads $k = \frac{1}{f(r)}\partial_t + \partial_r$ and $l = \frac{1}{2}\partial_t - \frac{f(r)}{2}\partial_r$, the four spin coefficients, $\kappa, \sigma, \lambda, \nu$ become zero. By Goldberg-Sachs theorem the spacetime is therefore Petrov type D and both its repeated principal null congruences are geodesic and sheer-free [22]. This solution is also asymptotically flat, however, due to the singularity along half of the symmetry axis, $\theta = \pi$, the solution cannot be globally asymptotically flat.

This singularity is quasi-regular singularity (as opposed to scalar curvature singularity), and it's interpretations by Lynden-Bell and Nouri-Zonoz (1998) advocate that there are small advances of perihelion in successive orbits (similar to what was observed by NUT for small l.) Hence although this is a purely gravitational solution, trajectories in this spacetime seem to describe the motion of a charged particle subjected to both a central force and a magnetic force due to a charged magnetic monopole at the origin, with mass 'm' and gravitomagnetic monopole (dyonic) moment 'l'. Bonnor (1960a) treats the $\theta = \pi$ singularity as a "frame dragging" caused by a thin semi-infinite spinning rod.

In 1963, Misner provided an alternative interpretation of Taub-NUT solution by forcing the axis to be completely regular. This is done by applying the transformation $t \to \tilde{t} = t - 4l\phi$ for the metric in the hemisphere $\pi/2 < \theta < \pi$ and gluing it to the reference metric (A.1) at $\theta = \pi/2$. Because ϕ is periodic, t coordinate has to be periodic with period $8\pi l$, which also ensures that the two coordinate patches join smoothly. This makes the spacetime topology to be homeomorphic to $R^1 \times S^3$ giving the time-dependent region a reasonable interpretation as a closed Taub universe which exists for a finite time. However, this happens at the expanse of requiring the stationary NUT regions to have closed timelike geodesics.

Including an additional discrete 2-space curvature parameter, ϵ with values $+1, 0$ and -1, the original Taub-NUT solution (A.1) can be extended to include an electric charge parameter e and a non-zero cosmological constant Λ,

$$g = -f(r)\left(\mathrm{d}t + l\frac{i(\zeta \mathrm{d}\bar{\zeta} - \bar{\zeta}\mathrm{d}\zeta)}{1 + \frac{1}{2}\epsilon\zeta\bar{\zeta}}\right)^2 + \frac{\mathrm{d}r^2}{f(r)} + (r^2 + l^2)\frac{2\mathrm{d}\zeta\mathrm{d}\bar{\zeta}}{(1 + \frac{1}{2}\epsilon\zeta\bar{\zeta})^2}, \tag{A.4}$$

where

$$f(r) = \frac{1}{r^2 + l^2}\left[\epsilon(r^2 - l^2) - 2mr + e^2 - \Lambda\left(\frac{1}{3}r^4 + 2l^2r^2 - l^4\right)\right]. \tag{A.5}$$

The major difference compared to (A.1) is that the numerator of $f(r)$ becomes quartic rather than quadratic, giving an additional pair of cosmological-type horizons.

Appendix B

Group theoretical aspects of non-linear σ-models

Let G be a non-compact real form of some-compact Lie group. There is an involutive automorphism $\tau : G \to G$, $\tau^2 = 1$, such that

$$H = \{h \in G : \tau(h) = h\} \tag{B.1}$$

is the maximal compact subgroup of G and the coset space G/H is a non-compact Riemannian symmetric space [42]. Let P be a representative element of G/H. Freedom to choose representatives leads to a gauge invariance with gauge group H, with the action of G on G/H,

$$P(x) \mapsto h(x)P(x)g^{-1}, \qquad h(x) \in H,\ g \in G \tag{B.2}$$

Because the field equations of P are invariant under the action of both G and H, the true value of the dynamical variables is in G/H. A metric γ on G/H can be defined as

$$\mathrm{d}\phi^i \mathrm{d}\phi^j \gamma_{ij}(\phi) = \langle \mathscr{I}, \mathscr{I} \rangle \tag{B.3}$$

where $\langle \cdot, \cdot \rangle$ is any invariant scalar product on the Lie algebra $\mathscr{G}$ of G and $\mathscr{I}$ is a G-invariant 1-form which transforms H-covariantly, i.e.,

$$\mathscr{I} = \frac{1}{2}\left(\mathrm{d}PP^{-1} + \tau(\mathrm{d}PP^{-1})\right) \mapsto h\mathscr{I}h^{-1} \tag{B.4}$$

The automorphism τ provides us with a natural candidate for $\mathscr{I}$ using the cannonical embedding of G/H in G [43],

$$P \mapsto M = \tau(P^{-1})P, \tau(M) = M^{-1} \tag{B.5}$$

where M is H-invariant and transforms covariantly under G. Then

$$\mathscr{I} = \mathcal{D}PP^{-1} = \frac{1}{2}PM^{-1}\mathrm{d}MP^{-1} \equiv PJP^{-1} \tag{B.6}$$

where $\mathcal{D}P$ is the H-covariant derivative of P and J is the current corresponding to M. Then (B.3) provides the metric on G/H and the σ-model field equations are

$$\mathcal{D}^\alpha \mathscr{I}_\alpha = 0 \tag{B.7}$$

where $\mathcal{D}_\alpha \mathscr{I}_\beta = \nabla_\alpha \mathscr{I}_\beta - [\mathscr{A}_\alpha, \mathscr{I}_\beta]$, where $\mathscr{A}$ is the connection for H. Hence we can conclude that the "metric" M is sufficient to formulate the σ-model. We can now give the precise definition of Geroch group:

The Geroch group

Geroch group is the central extension $G^{(\infty)}_{\mathrm{ce}}$ of a group of holomorphic functions $G^{(\infty)}$ with values in $SL(2,\mathbb{R})$ generated by transformations of λ, the conformal factor. It acts in the usual non-linear way on infinite dimensional coset space $G^{(\infty)}_{\mathrm{ce}}/H^{(\infty)}$ of the non-linear sigma-model.

Appendix C

Kaluza-Klein theory

Consider 5 dimensional Kaluza-Klein theory, with metric $\hat{g}_{MN}$ of signature {+, -, -, -, -}. Upper-case latin indices (M,N,... = 0, 1, .., 5) represent coordinates on V^5 and Greek indices ($\mu, \nu, ... = 0, 1, .., 4$) denote coordinates on V^4. The action is then given by

$$I_G = -\frac{1}{2\hat{\kappa}} \int d^5 z \sqrt{\hat{g}} \hat{R} \tag{C.1}$$

where ($z^M = x^\mu, y$) and ($\hat{.}$) denotes quantities on V^5. To ensure that the fifth dimension is not observed, we assume the "cylinder condition",

$$\partial_y \hat{g}_{MN} = 0 \tag{C.2}$$

Eq. (C.2) is not invariant under general coordinate transformation. However, it is invariant under transformations of the form

$$x'^\mu = x'^\mu(x), \quad y' = \rho y + \epsilon(x) \tag{C.3}$$

where ρ is a constant (this can be shown by demanding that eq. (C.2) holds true after a coordinate transformation, where $\hat{g}_{MN}$ behaves like a usual rank-2 tensor). We can write the metric on V^5 as

$$\hat{g}^{(5)}_{\mu\nu} = \left[\begin{array}{c|c} g^{(4)}_{\mu\nu} & g_{\mu 5} \\ \hline g_{5\nu} & g_{55} \end{array}\right] \tag{C.4}$$

In the following we will assume that $g_{\mu 5} = e_\mu e_5 \neq 0$. Let us decompose the vielbein into a component parallel and perpendicular to e_5, i.e., $e_\mu = e_{\mu_\perp} + e_{\mu_\parallel}$ such that $e_{\mu_\perp}.e_5 = 0$. Therefore, $e_{\mu_\parallel} = \frac{g_{\mu 5}}{g_{55}} e_5$ and hence the 5 dimensional metric can be decomposed into

$$\hat{g}^{(5)}_{MN} = g^{(4)}_{MN} + \frac{B_M B_N}{\Phi} \tag{C.5}$$

where $B_M = g_{M5}/g_{55}$ and $\Phi = g_{55}$. Substituting (C.5) in (C.3) with $\rho = 1$, we can show that B_μ transforms as

$$B'_\mu = B_\mu + \partial_\mu \epsilon \tag{C.6}$$

This is very similar to the gauge property of electromagnetic potential. In fact, if we assume that $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$, then one can show that the action (C.1) can be decomposed as

$$I^{(0)}_G = -\frac{1}{2\kappa} \int d^4 x \sqrt{-g} \left(R + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) = S_{EH} + S_{EM} \tag{C.7}$$

where we take $g_{55} = -1$ and normalized B_μ such that $B_\mu = \sqrt{2\kappa} A_\mu$

There are 2 ways in which we can understand these results: 1. Using cylinder condition, all that remains from the use of the fifth dimension is the increased number of fields in d = 4, the reduced four-dimensional theory describes gravity and electromagnetism. This procedure is called dimensional reduction, 2. Or, we can accept the fifth dimension as a physical reality, and obtain all geometric properties of V^5. This approach, based on the modern concept of spontaneous symmetry breaking, is known as spontaneous compactification.

Appendix D

Plebański-Demiański family

The Plebański-Demiański (PD) form of the family of type D solutions with two commuting Killing vectors is given by

$$\mathrm{d}s^2 = \frac{1}{(1-\hat{p}\hat{r})^2}\left[-\frac{\hat{Q}(\mathrm{d}\hat{\tau}-\hat{p}^2\mathrm{d}\hat{\sigma})^2}{\hat{r}^2+\hat{p}^2}+\frac{\hat{P}(\mathrm{d}\hat{\tau}+\hat{r}^2\mathrm{d}\hat{\sigma})^2}{\hat{r}^2+\hat{p}^2}+\frac{\hat{r}^2+\hat{p}^2}{\hat{P}}\mathrm{d}\hat{p}^2+\frac{\hat{r}^2+\hat{p}^2}{\hat{Q}}\mathrm{d}\hat{r}^2\right] \tag{D.1}$$

for two quartic functions

$$\begin{aligned}\hat{P}(\hat{p}) &= \hat{k}+2\hat{n}\hat{p}-\hat{\epsilon}\hat{p}^2+2\hat{m}\hat{p}^3-(\hat{k}+\hat{e}^2+\hat{g}^2+\Lambda/3)\hat{p}^4\\ \hat{Q}(\hat{r}) &= (\hat{k}+\hat{e}^2+\hat{g}^2)-2\hat{m}\hat{r}+\hat{\epsilon}\hat{r}^2-2\hat{n}\hat{r}^3-(\hat{k}+\Lambda/3)\hat{r}^4\end{aligned} \tag{D.2}$$

with $\hat{m}, \hat{n}, \hat{e}$, and $\hat{g}$ representing the mass, nut charge, electric charge and magnetic charge respectively in the appropriate subclasses of PD family. Furthermore, with the addition of two other continuous parameters α and ω, we can set $\hat{\epsilon}$ and $\hat{k}$ to $+1, 0$ or -1, where α and ω are related to acceleration and rotation in appropriate subclasses.

Appendix E

Relativistic hydrodynamics

Ideal fluid

For a relativistic fluid, mass and energy are on the same footing and hence instead of working with mass density, we should work with energy density, $\epsilon(t, \vec{x})$ and similarly with 4-velocity $u^\mu = dx^\mu/d\tau$. We can write the normalized 4-velocity as [44]

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{dt}{d\tau}\frac{dx^\mu}{dt} = \gamma(\vec{v})\begin{pmatrix}1\\ \vec{v}\end{pmatrix} \tag{E.1}$$

Note that due to the normalization, 4-velocity u^μ has the same information as the three-velocity. The energy momentum tensor for an ideal fluid has to be built out of the hydrodynamic degrees of freedoms. The symmetry properties of EM tensor imply that in the local rest frame of the fluid (where velocity and the direction of energy flow are aligned),

$$T_{(0)} = (\epsilon + p)\frac{u_\mu u_\nu}{k^2} + p g_{\mu\nu} \tag{E.2}$$

where we have normalized the velocity as $||u||^2 = -k^2$, where k is the speed of light. Since u^μ is along the temporal direction, we can decompose the spacetime into spatial slices using a projector (i.e., the induced metric)

$$h_{\mu\nu} = g_{\mu\nu} + \frac{u_\mu u_\nu}{k^2} \tag{E.3}$$

When there are no external sources, the energy-momentum is conserved

$$\nabla_a T^{ab}_{(0)} = 0, \tag{E.4}$$

and the conformal equation of state becomes $\epsilon = 2p$.

Dissipative fluid

To study the system away from equilibrium, we need to add a dissipative term to the stress tensor. Since hydrodynamics is an effective field theory description, we can use the standard QFT techniques and construct the effective lagrangian by considering all possible relevant operators consistent with the symmetries. Since in this case the only effective dynamical variables of the system are temperature and velocity field (see the summary of 4.1), the relevant operators are built using $T(x)$, $u(x)$ and their derivatives. However, before we build these operators, we have to remember that for relativistic fluids, the exact meaning of 'velocity of fluid' becomes questionable (because of the equivalence of mass and energy, i.e., between heat flux and energy flow). This ambiguity will allow us to choose a velocity field (see F.3). In the local rest frame of a fluid element, we can decompose the dissipation term into components orthogonal to the velocity field and components along it, i.e., if the stress tensor is

$$T = T_{(0)} + T_{dis} \tag{E.5}$$

then $T^{\mu\nu}_{dis} = \pi^{\mu\nu} - u^\mu q^\nu$, for some vectors $\pi^{\mu\nu}$ and q^ν, such that

$$u_\mu \pi^{\mu\nu} = 0 \tag{E.6}$$

Furthermore, using (E.4), we can relate the gradients of pressure and energy density with the derivatives of u,

$$\begin{aligned} u^\nu \nabla_\mu T^{\mu\nu}_{(0)} = 0 \quad &\Longrightarrow \quad (\epsilon + p)\nabla_\mu u^\mu + u^\mu \nabla_\mu \epsilon = 0 \\ h_{\nu\alpha} \nabla_\mu T^{\mu\nu}_{(0)} = 0 \quad &\Longrightarrow \quad h_\alpha{}^\mu \nabla_\mu \epsilon + (\epsilon + p) h_{\nu\alpha} u^\mu \nabla_\mu u^\nu = 0 \end{aligned} \tag{E.7}$$

Let us decompose the gradient of u^μ as

$$\nabla_\mu u_\mu = \frac{1}{d-1}\Theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} - q_\mu u_\nu \tag{E.8}$$

where up to leading order,

$$\begin{aligned} a_\mu &= u^\nu \nabla_\nu u_\mu \\ \Theta &= \nabla_\mu u^\mu \\ \sigma_{\mu\nu} &= \nabla_{(\mu} u_{\nu)} + \frac{1}{k^2} u_{(\mu} a_{\nu)} - \frac{1}{2}\Theta h_{\mu\nu} \\ \omega_{\mu\nu} &= \nabla_{[\mu} u_{\nu]} + \frac{1}{k^2} u_{[\mu} a_{\nu]} \end{aligned} \tag{E.9}$$

are respectively the acceleration, the expansion, the shear and the vorticity of the congruence u. Since at leading order the dissipative part of the stress tensor is also built from gradient of u^μ, (E.6) and (E.8) tells us that the only symmetric rank-2 tensor is

$$T_{\mu\nu} = (\epsilon + p)\frac{u_\mu u_\nu}{k^2} + p g_{\mu\nu} + \tau_{\mu\nu} + \frac{u_\mu q_\nu}{k^2} + \frac{u_\nu q_\mu}{k^2} \tag{E.10}$$

Equation (E.6) then enforces that

$$u^\mu \tau_{\mu\nu} = 0, \qquad u^\mu q_\mu = 0 \tag{E.11}$$

Upto first-order in hydrodynamics, the boundary heat current q and viscous stress tensor τ can be expressed as expansions in fluid velocity and temperature derivatives,

$$\tau_{(1)\mu\nu} = -2\eta\sigma_{\mu\nu} - \zeta h_{\mu\nu}\Theta, \tag{E.12}$$

$$q_{(1)\mu} = -\kappa h_\mu{}^\nu \left(\partial_\nu T + \frac{T}{k^2} a_\nu\right), \tag{E.13}$$

where η, ζ are the shear and bulk viscosities, and κ is the thermal conductivity. We also define the scalar vorticity γ,

$$\gamma^2 = \frac{1}{2k^4}\omega_{\mu\nu}\omega^{\mu\nu}. \tag{E.14}$$

and the Hodge dual to vorticity two-form as:

$$\star\,\omega = k\gamma u \tag{E.15}$$

Appendix F

Fluid/gravity correspondence

The holographic principle states that there exists a one-to-one map between the single particle states in classical Hilbert space of gravitational theory and single trace operators in gauge theory. The most familiar example of this is the duality between $SU(N)$ $\mathcal{N}=4$ super Yang-Mills (SYM) theory and type-IIB string theory on $\mathrm{AdS}_5 \times S^5$. Interestingly on the string theory side, at large coupling λ, type-IIB supergravity on $\mathrm{AdS}_5 \times S^5$ admits a universal truncation to Einstein's gravity with negative cosmological constant [24],

$$E_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \qquad \Lambda \equiv -\frac{d(d-1)}{2 l_{\mathrm{AdS}}^2} \tag{F.1}$$

This suggests that there exists a decoupled sector of stress tensor dynamics in $\mathcal{N}=4$ SYM at large λ. In fact, at two-derivative level, there is an infinite number of conformal gauge theories for which (F.1) describes the universal decoupled dynamics of the stress tensor. The dynamical equation of hydrodynamics, conservation of stress tensor (E.4), depends only on stress tensor and hence should lie in this decoupled sector. This inspired the authors of [24–29] to study the AdS/CFT correspondence in the hydrodynamic limit (large temperature and long distances w.r.t. l_{AdS}) which lead to a map between fluid dynamics and gravity, now known as the fluid/gravity correspondence.

The correspondence can be summarized as follows:

	Bulk		Boundary
At eq^m	Static black holes	$\longleftrightarrow$	Static frame with a temp. field T(x)
	Hawking temperature	$\longleftrightarrow$	Temperature of the fluid
	Horizon boost velocity	$\longleftrightarrow$	Fluid dynamical velocity

Away from the equilibrium, the solutions are corrected order by order in a derivative expansion. Then the domains of "nearly constant fluid variables" on the boundary can be extended radially into the bulk [24]. Then,

Away from eq^m	Patched black holes with temperature gradient, s.t., they obey (F.1)	$\longleftrightarrow$	Boundary fluid with varying velocity and temperature
		At leading order:	
	Radial momentum constraints for gravity in AdS	$\longleftrightarrow$	Velocity and temp. fields obey ideal fluid dynamics, (E.4)

Boundary metric

Let the boundary be a D-dimensional hyperbolic geometry equipped with a boundary metric

$$ds^2 = g_{MN} dx^M dx^N \tag{F.2}$$

Let u^μ be the velocity of the boundary fluid with its acceleration, expansion, shear and vorticity given by (E.9). If t is the coordinate adapted to u, i.e., $\overline{u} = \partial_t/\Omega$, then

$$\underline{u} = -k^2(\Omega \mathrm{d}t - b_i \mathrm{d}x^i) \tag{F.3}$$

and in this choice of coordinates, $h_{0M} = 0$ (where $h_{\mu\nu}$ is taken from (E.3), and is w.r.t. the three-dimensional metric (F.2)). Hence the metric becomes

$$\mathrm{d}s^2_{bry} = -k^2(\Omega \mathrm{d}t - b_i \mathrm{d}x^i)^2 + a_{ij}\mathrm{d}x^i \mathrm{d}x^j \tag{F.4}$$

which is also referred to as Randers-Papapetrou metric. In this adapted frame (F.4), the Weyl connection, and various parameters of the congruence u (E.9) are given by

$$\mathrm{A} = \frac{1}{\Omega}\left(\partial_i \Omega + \partial_t b_i - \frac{1}{2} b_i \partial_t \ln \sqrt{a}\right) \mathrm{d}x^i + \frac{1}{2}\partial_t \ln \sqrt{a}\mathrm{d}t\,, \tag{F.5}$$

$$\begin{aligned}
a &= k^2\left(\partial_i \ln \Omega + \frac{1}{\Omega}\partial_t b_i\right)\mathrm{d}x^i,\\
\omega &= \frac{k^2}{2}\left(\partial_i b_j + \frac{1}{\Omega} b_i \partial_j \Omega + \frac{1}{\Omega} b_i \partial_t b_j\right)\mathrm{d}x^i \wedge \mathrm{d}x^j,\\
\Theta &= \frac{1}{\Omega}\partial_t \ln \sqrt{a},\\
\sigma &= \frac{1}{2\Omega}\left(\partial_t a_{ij} - a_{ij}\partial_t \ln \sqrt{a}\right)\mathrm{d}x^i \mathrm{d}x^j.
\end{aligned} \tag{F.6}$$

When ∂_t is a Killing field, the shear in (F.6) vanishes and we can parametrize the metric as [45]

$$\mathrm{d}s^2_{\mathrm{bdry}} = -\Omega^2(\mathrm{d}t - \vec{b})^2 + \mathrm{d}l^2 = -\Omega^2(\mathrm{d}t - \vec{b})^2 + \frac{2}{\kappa^2 P^2}\mathrm{d}\zeta \mathrm{d}\bar{\zeta} \tag{F.7}$$

where P and Ω are arbitrary real functions of $(t, \zeta, \bar{\zeta})$ and

$$\vec{b} = B(t, \zeta, \bar{\zeta})\mathrm{d}\zeta + \bar{B}(t, \zeta, \bar{\zeta})\mathrm{d}\bar{\zeta} \tag{F.8}$$

Derivative expansion

With this boundary metric, we can now construct the bulk geometry. The bulk diffeomorphisms correspond to conformal rescalings of boundary metric $\mathrm{d}s^2_{\mathrm{bdry}} \to \mathrm{d}s^2_{\mathrm{bdry}}/\mathcal{B}^2$, and hence bulk metric should be insensitive them. Covariantization with respect to the rescalings then require that, $\mathscr{D}_\mu u^\mu$ and $u^\lambda \mathscr{D}_\lambda u_\mu$ should vanish, where we have replaced the ordinary derivatives by Weyl covariant ones, i.e., for a tensor v_μ of weight-w,

$$\mathscr{D}_\nu v_\mu = \nabla_\nu v_\mu + (w+1)A_\nu v_\mu + A_\mu v_\nu - g_{\mu\nu}A^\rho v_\rho \tag{F.9}$$

where A is Weyl connection one-form

$$\mathrm{A} = \frac{1}{k^2}\left(\mathrm{a} - \frac{\Theta}{2}u\right) \tag{F.10}$$

that transforms as $A \to A -\ \ln \mathcal{B}$. For any symmetric and traceless tensor of conformal weight 1 (like EM or Cotton tensor), $\mathscr{D}_\nu S^\nu{}_\mu = \nabla_\nu S^\nu{}_\mu$. Correspondingly, we define the

Weyl covariantized Riemann tensor$\mathscr{R}_{\mu\nu\rho}{}^{\sigma}$, Ricci tensor $\mathscr{R}_{\mu\nu}$ etc. In our notation, Weyl covariant quantities will be denoted by curved letters like $\mathscr{A}\mathscr{B}\mathscr{C}$.

The authors of [9–13] then showed that for a sheerless boundary fluid, resummed bulk metric is given by

$$\begin{aligned} \mathrm{ds_{bulk}}^2 &= 2\frac{u}{k^2}(\mathrm{dr}+rA)+r^2\mathrm{ds}^2+\frac{\mathrm{S}}{k^4} \\ &+\frac{u^2}{k^4r^2}\left(1-\frac{1}{2k^4r^2}\omega_{\alpha\beta}\omega^{\alpha\beta}\right)\left(\frac{8\pi GT_{\lambda\mu}u^\lambda u^\mu}{k^2}r+\frac{C_{\lambda\mu}u^\lambda\eta^{\mu\nu\sigma}\omega_{\nu\sigma}}{2k^4}\right) \\ &+\text{terms involving the shear }\sigma+O(\mathscr{D}^4u) \end{aligned} \tag{F.11}$$

Here, r is the radial coordinate, x^μ are the three boundary coordinates, k is a constant related to the cosmological constant by $\Lambda=-3k^2$ and $\kappa=3k/8\pi G$, while S is a Weyl-invariant tensor related to the Schouten tensor, $\mathscr{S}_{\nu\sigma}=\mathscr{R}_{\nu\sigma}-\frac{1}{4}\mathscr{R}g_{\nu\sigma}$ and is given by

$$\mathrm{S}=S_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu=-2\mathrm{u}\mathscr{D}_\nu\omega^\nu{}_\mu\mathrm{d}x^\mu-\omega_\mu{}^\lambda\omega_{\lambda\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu-\mathrm{u}^2\frac{\mathscr{R}}{2}; \tag{F.12}$$

The metric can be resummed into the closed form (4.6), by identifying the resummation coordinate ρ, such that

$$\rho^2=r^2+\gamma^2=r^2+\frac{q^2}{4k^4} \tag{F.13}$$

where γ was defined in (E.14). The derivative expansion can be organized in the powers of q, which is the proportionality constant of the hodge dual of vorticity with velocity field[1], i.e.,

$$\eta^{\mu\rho\sigma}\omega_{\nu\sigma}=qu^\mu \tag{F.14}$$

In presence of fluid congruence u, this allows us to define a fully antisymmetric two-index tensor,

$$\eta_{\mu\nu}=-\frac{u^\rho}{k}\eta_{\rho\mu\nu}, \tag{F.15}$$

obeying

$$\eta_{\mu\sigma}\eta_\nu{}^\sigma=h_{\mu\nu} \tag{F.16}$$

where $h_{\mu\nu}$ is w.r.t. the bulk metric (F.11). This implies that γ can be further simplified to give

$$\gamma^2=\frac{1}{2}a^{ik}a^{jl}\left(\partial_{[i}b_{j]}+\frac{1}{\Omega}b_{[i}\partial_{j]}\Omega+\frac{1}{\Omega}b_{[i}\partial_t b_{j]}\right)\left(\partial_{[k}b_{l]}+\frac{1}{\Omega}b_{[k}\partial_{l]}\Omega+\frac{1}{\Omega}b_{[k}\partial_t b_{l]}\right) \tag{F.17}$$

In three dimensions the Weyl tensor is identically zero and all the information about geometry is captured in symmetric traceless conserved Cotton tensor $C=C_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$, given by

$$C_{\mu\nu}=\eta_\mu{}^{\rho\sigma}\mathscr{D}_\rho\left(\mathscr{S}_{\nu\sigma}+F_{\nu\sigma}\right)=\eta_\mu{}^{\rho\sigma}\nabla_\rho\left(R_{\nu\sigma}-\frac{R}{4}g_{\nu\sigma}\right) \tag{F.18}$$

where $F_{\nu\sigma}=\partial_\nu A_\sigma-\partial_\sigma A_\nu$ is the antisymmetric field strength tensor and can be thought of as the effective torsion,

$$\left(\mathscr{D}_\mu\mathscr{D}_\nu-\mathscr{D}_\nu\mathscr{D}_\mu\right)f=wfF_{\mu\nu} \tag{F.19}$$

[1] where η is similar to the one obtained in section 1.1 with $\eta^{\mu\rho\sigma}=\epsilon^{\mu\rho\sigma}/\sqrt{-g}$

This tensor is vanishing iff spacetime is conformally flat. The Weyl covariant Riemann tensor and Ricci tensor are defined as,

$$\left(\mathscr{D}_\mu \mathscr{D}_\nu - \mathscr{D}_\nu \mathscr{D}_\mu\right) V^\rho = \mathscr{R}^\rho{}_{\sigma\mu\nu} V^\sigma + w V^\rho F_{\mu\nu} \tag{F.20}$$

$$\mathscr{R}_{\mu\nu} = R_{\mu\nu} + \nabla_\nu A_\mu + A_\mu A_\nu + g_{\mu\nu}\left(\nabla_\lambda A^\lambda - A_\lambda A^\lambda\right) - F_{\mu\nu}, \tag{F.21}$$

where V^ρ are weight-w. In Landau frame, hydrodynamic components of $\pi_{\mu\nu}$ are transverse to u because of (E.6) and the boundary energy and cotton density are introduced as $\epsilon(x) = T_{\mu\nu} u^\mu u^\nu$ and $c(x) = C_{\mu\nu} u^\mu u^\nu$. Vorticity of the boundary congruence ω, is given by $\frac{1}{2}\omega_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$. Cotton density, current and stress tensor can be further expressed as

$$c = \frac{1}{k^3} C_{\mu\nu} u^\mu u^\nu = \frac{1}{k^2} u^\nu \eta^{\sigma\rho} \mathscr{D}_\rho \left(\mathscr{S}_{\nu\sigma} + F_{\nu\sigma}\right), \tag{F.22}$$

$$c_\nu = -c u_\nu - \frac{u^\mu C_{\mu\nu}}{k} = \eta^{\rho\sigma} \mathscr{D}_\rho \left(\mathscr{S}_{\nu\sigma} + F_{\nu\sigma}\right) - c u_\nu, \tag{F.23}$$

$$c_{\mu\nu} = -k h^\rho{}_\mu h^\sigma{}_\nu C_{\rho\sigma} + \frac{ck^2}{2} h_{\mu\nu} = -h^\lambda{}_\mu \left(k \eta_\nu{}^{\rho\sigma} - u_\nu \eta^{\rho\sigma}\right) \mathscr{D}_\rho \left(\mathscr{S}_{\lambda\sigma} + F_{\lambda\sigma}\right) + \frac{ck^2}{2} h_{\mu\nu} \tag{F.24}$$

With the above boundary data, the bulk velocity field ∂_r of the 4D resummed metric (4.6) is null, geodesic and shear-free [10]. From the corollary of Goldberg-Sachs theorem [22], the Einstein bulk geometry (4.6) is therefore algebraically special.

Appendix G

Flatspace holography

In section 4.2.1 we showed that Zermelo's metric is the natural relativistic spacetime uplift of a Galilean space endowed with Newton–Cartan structure. In this apppendix we will first show that Rander-Papapetrou metric is the natural "ultra-relativistic" uplift of Carrollian space and then go on to motivate flat holography. We will also show the explicit expressions for some of the quantities discussed in section 4.2.

Similar to the Galilean case (4.14) and (4.15), we can define an action which is invariant under the Carrollian diffeomorphisms, $t' = t'(t, \mathbf{x})$ & $\mathbf{x}' = \mathbf{x}'(\mathbf{x})$:

$$S[t] = \int_{\mathscr{V}\subset\mathscr{S}} \mathrm{d}^d x \sqrt{a} \mathcal{L}(\boldsymbol{\partial} t, t, \mathbf{x}), \tag{G.1}$$

where $\mathcal{L}(\boldsymbol{\partial} t, t, \mathbf{x})$ is the Lagrangian density:

$$\mathcal{L}(\boldsymbol{\partial} t, t, \mathbf{x}) = \frac{1}{2} a^{ij} \left(\Omega \partial_i t - b_i\right) \left(\Omega \partial_j t - b_j\right). \tag{G.2}$$

associated to fields on a d-dimensional Euclidean space $\mathcal{S}$ given by (4.13), and equipped with real time line $t \in \mathbb{R}$. Here b_i represents the inverse velocity defined in footnote 6, which transforms under Carrollian diffeomorphisms as

$$b'_k = \left(b_i + \frac{\Omega}{J} j_i\right) J^{-1 i}{}_k. \tag{G.3}$$

where the Jacobian functions J, j_i and J^i_j are defined as

$$J(t, \mathbf{x}) = \frac{\partial t'}{\partial t}, \quad j_i(t, \mathbf{x}) = \frac{\partial t'}{\partial x^i}, \quad J^i_j(\mathbf{x}) = \frac{\partial x^{i\prime}}{\partial x^j}. \tag{G.4}$$

Similar to section 4.2.1, we can show that by considering a relativistic instantonic d-brane in a pseudo-Riemannian manifold $\mathcal{M}$, endowed with Randers-Papapetrou metric (F.4), the Dirac–Born–Infeld action reads:

$$S[h] = \int_{\mathscr{V}} \mathrm{d}^d y \sqrt{h}, \tag{G.5}$$

where h is the determinant of the induced metric matrix

$$h_{ij} = g_{\mu\nu} \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\nu}{\partial y^j} \tag{G.6}$$

with $g_{\mu\nu}$ given by (F.4). We can compute this determinant h in powers of c^2 to give

$$\sqrt{h} = \det \alpha \sqrt{a} \left(1 - \frac{c^2}{2} a^{kl} \left(\Omega \partial_k t - b_k\right) \left(\Omega \partial_l t - b_l\right) + \mathrm{O}\left(c^4\right)\right). \tag{G.7}$$

Hence (G.5) becomes

$$S[h] = \int_{\mathscr{V}} \mathrm{d}^d x \sqrt{a} \left(1 - \frac{c^2}{2} a^{kl} \left(\Omega \partial_k t - b_k\right) \left(\Omega \partial_l t - b_l\right) + \mathrm{O}\left(c^4\right)\right). \tag{G.8}$$

Neglecting the constant (Carrollian invariant) term, (G.8) describes the same dynamics as (G.1), (G.2) proving our claim that Randers-Papapetrou metric (4.3) is the natural "ultra-relativistic" uplift of Carrollian space endowed with a Carrollian structure.

Also note that under Carrollian diffeomorphisms, the ordinary exterior derivatives do not transform as a form, i.e.,

$$\partial_t' = \frac{1}{J}\partial_t, \quad \partial_j' = J^{-1}{}^i_j\left(\partial_i - \frac{j_i}{J}\partial_t\right). \tag{G.9}$$

While the time derivative transforms appropriately, its action on metric does not vanish, i.e., $\partial_t a_{ij} \neq 0$. This motivates us to define new Carroll-covarint derivatives,

$$\begin{aligned}\frac{1}{\Omega}\hat{\partial}_t V^i &= \frac{1}{\Omega}\partial_t V^i + \hat{\gamma}^i{}_j V^j \\ \hat{\partial}_i &= \partial_i + \frac{b_i}{\Omega}\partial_t\end{aligned} \tag{G.10}$$

which indeed transforms as

$$\hat{\partial}_i' = J^{-1}{}^j_i\hat{\partial}_j. \tag{G.11}$$

under Carrollian diffeomorphisms. When this derivative acts on scalars, we get a one-form, however, for other tensors we need to introduce a new *Levi–Civita–Carroll* connection $\hat{\gamma}^i_{jk} = \gamma^i_{jk} + c^i_{jk}$ where γ^i_{jk} is the standard Levi-Civita connection. The $\hat{\gamma}^i{}_j$ appearing in (G.10) is the corresponding "temporal Carrollian connection"

$$\hat{\gamma}^i{}_j = \frac{1}{2\Omega}a^{ik}\partial_t a_{kj}, \tag{G.12}$$

Note that on scalar functions the Carrollian time derivative acts as an ordinary time derivative: $\hat{\partial}_t\Phi = \partial_t\Phi$. The trace and traceless parts of (G.12) are identified with the Carrollian expansion and shear. Hence the parameters associated with Carrollian fluids are:

$$\begin{aligned}\theta &= \hat{\gamma}^i{}_i = \frac{1}{\Omega}\partial_t \ln\sqrt{a}, \\ \xi^i{}_j &= \hat{\gamma}^i{}_j - \frac{1}{2}\delta^i_j\theta = \frac{1}{2\Omega}a^{ik}\left(\partial_t a_{kj} - a_{kj}\partial_t\ln\sqrt{a}\right), \\ \varphi_i &= \frac{1}{\Omega}\left(\partial_t b_i + \partial_i\Omega\right) = \partial_t\frac{b_i}{\Omega} + \hat{\partial}_i\ln\Omega, \\ \varpi_{ij} &= \partial_{[i}b_{j]} + b_{[i}\varphi_{j]} = \frac{\Omega}{2}\left(\hat{\partial}_i\frac{b_j}{\Omega} - \hat{\partial}_j\frac{b_i}{\Omega}\right)\end{aligned} \tag{G.13}$$

We can also define the corresponding Levi-Civita covariant derivatives $\hat{\nabla} = \hat{\partial} + \hat{\gamma}$ and the Riemann-Carroll and Ricci-Carroll tensors:

$$\hat{r}^i{}_{jkl} = \hat{\partial}_k\hat{\gamma}^i_{lj} - \hat{\partial}_l\hat{\gamma}^i_{kj} + \hat{\gamma}^i_{km}\hat{\gamma}^m_{lj} - \hat{\gamma}^i_{lm}\hat{\gamma}^m_{kj}, \quad \text{and} \tag{G.14}$$

$$\begin{aligned}\hat{r}_{ij} &= \hat{r}^k{}_{ikj} \\ &= \hat{s}_{ij} + \hat{K}a_{ij} + \hat{A}\eta_{ij}.\end{aligned} \tag{G.15}$$

Ricci-Carroll tensor is in general not symmetric and hence we have defined $\hat{s}_{ij}$ as the symmetric and traceless part which vanishes when the Carrollian fluid is shearless and,

$$\hat{K} = \frac{1}{2}a^{ij}\hat{r}_{ij} = \frac{1}{2}\hat{r}, \quad \hat{A} = \frac{1}{2}\eta^{ij}\hat{r}_{ij} = *\varpi\theta \tag{G.16}$$

are the scalar-electric and scalar-magnetic Gauss–Carroll curvatures, with

$$*\,\varpi = \frac{1}{2}\eta^{ij}\varpi_{ij}. \tag{G.17}$$

Demanding Weyl invariance of the boundary Carrollian data[1] is equivalent to defining a Weyl connection $\alpha_i = \varphi_i - \frac{\theta}{2}b_i$ which transforms under Weyl transformations as

$$\alpha_i \to \alpha_i - \partial_i \ln \mathcal{B}. \tag{G.18}$$

where $\mathcal{B} = \mathcal{B}(t, \mathbf{x})$ is an arbitrary function, s.t. the boundary geometry transforms as

$$a_{ij} \to \frac{a_{ij}}{\mathcal{B}^2}, \quad b_i \to \frac{b_i}{\mathcal{B}}, \quad \Omega \to \frac{\Omega}{\mathcal{B}}, \tag{G.19}$$

Corresponding to this Weyl-connection, we define Weyl-Carroll covariant derivatives that $\hat{\mathscr{D}}_i$ and $\hat{\mathscr{D}}_t$ that obey

$$\hat{\mathscr{D}}_j a_{kl} = 0, \quad \hat{\mathscr{D}}_t a_{kl} = 0. \tag{G.20}$$

For a weight-w scalar function Φ, or a weight-w vector V^i, *i.e.*, one that scales as $\mathcal{B}^w$ under (G.19),

$$\hat{\mathscr{D}}_j \Phi = \hat{\partial}_j \Phi + w\varphi_j \Phi, \quad \hat{\mathscr{D}}_j V^l = \hat{\nabla}_j V^l + (w-1)\varphi_j V^l + \varphi^l V_j - \delta^l_j V^i \varphi_i, \tag{G.21}$$

Similarly, we define

$$\frac{1}{\Omega}\hat{\mathscr{D}}_t \Phi = \frac{1}{\Omega}\hat{\partial}_t \Phi + \frac{w}{2}\theta\Phi = \frac{1}{\Omega}\partial_t \Phi + \frac{w}{2}\theta\Phi, \tag{G.22}$$

and

$$\frac{1}{\Omega}\hat{\mathscr{D}}_t V^l = \frac{1}{\Omega}\hat{\partial}_t V^l + \frac{w-1}{2}\theta V^l = \frac{1}{\Omega}\partial_t V^l + \frac{w}{2}\theta V^l + \xi^l{}_i V^i, \tag{G.23}$$

where $\hat{\mathscr{D}}_i$ leaves the weight unaltered and $\frac{1}{\Omega}\hat{\mathscr{D}}_t$ increases the weight by one unit. We can the define,

$$\begin{aligned}\hat{\mathscr{R}}^i{}_{jkl} \;=\;& \hat{r}^i{}_{jkl} - \delta^i_j \varphi_{kl} - a_{jk}\hat{\nabla}_l \varphi^i + a_{jl}\hat{\nabla}_k \varphi^i + \delta^i_k \hat{\nabla}_l \varphi_j - \delta^i_l \hat{\nabla}_k \varphi_j \\ & + \varphi^i \left(\varphi_k a_{jl} - \varphi_l a_{jk}\right) - \left(\delta^i_k a_{jl} - \delta^i_l a_{jk}\right)\varphi_m \varphi^m + \left(\delta^i_k \varphi_l - \delta^i_l \varphi_k\right)\varphi_j \end{aligned} \tag{G.24}$$

as the Riemann–Weyl–Carroll weight-0 tensor, where $\varphi_{ij} = \hat{\partial}_i \varphi_j - \hat{\partial}_j \varphi_i$ is a Carrollian two-form. From this we can define the Ricci tensor,

$$\begin{aligned}\hat{\mathscr{R}}_{ij} &= \hat{\mathscr{R}}^k{}_{ikj} = \hat{r}_{ij} + a_{ij}\hat{\nabla}_k \varphi^k - \varphi_{ij}, \\ &= \hat{s}_{ij} + \hat{\mathscr{K}} a_{ij} + \hat{\mathscr{A}}\eta_{ij}.\end{aligned} \tag{G.25}$$

In this expression $\hat{s}_{ij}$ is a symmetric and traceless tensor, which vanishes for shearless Carrollian fluids. Also,

$$\hat{\mathscr{K}} = \frac{1}{2}a^{ij}\hat{\mathscr{R}}_{ij} = \hat{K} + \hat{\nabla}_k \varphi^k, \quad \hat{\mathscr{A}} = \frac{1}{2}\eta^{ij}\hat{\mathscr{R}}_{ij} = \hat{A} - *\varphi \tag{G.26}$$

[1]For flat case, the boundary is a spatial surface at null infinity $\mathscr{I}^+$. The geometry of $\mathscr{I}^+$ is equipped with a conformal class of metrics rather than with a unique metric. From a representative of this class, we must be able to explore others by Weyl transformations, and this amounts to studying conformal Carrollian geometry as opposed to plain Carrollian geometry

are the Weyl-Carroll scalar-electric and scalar-magnetic Gauss–Carroll curvatures both of weight 2. With all this data we are now in a position to take the vanishing-k limit of the resummed metric (F.11) and the boundary fluid data.

For the boundary data, let us assume that a_{ij}, b_i and Ω were independent of k and hence remain same in the vanishing-k limit. These then coupled with the integrability conditions of section 4.2 provide all the necessary Carrollian fluid data. Defining a Weyl invariant tensor,

$$\boldsymbol{s} = 2(\Omega \mathrm{d}t - \boldsymbol{b})\mathrm{d}x^i \eta^j{}_i \hat{\mathscr{D}}_j \star \omega + \star\omega^2 a_{ij} x^i \mathrm{d}x^j - \hat{\mathscr{K}}(\Omega \mathrm{d}t - \boldsymbol{b})^2 \tag{G.27}$$

we can show that the vanishing-k limit of the resummed metric (4.6), for algebraically special bulk geometries gives (4.27). The bulk metric is indeed Weyl invariant provided that r transforms a weight-1 tensor, i.e., (G.19) can be absorbed by setting $r \to \mathcal{B}r$. The integrability conditions are as outlined in 4.2.2 where all the information is encoded in $\mathrm{d}\ell^2$, b_i and Ω, such that: the Carrollian density,

$$c = \left(\hat{\mathscr{D}}_l \hat{\mathscr{D}}^l + 2\hat{\mathscr{K}}\right) * \varpi. \tag{G.28}$$

and the weight-2 current forms, χ_i and ψ_i from (4.21), and the weight-1 symmetric and traceless rank-two tensors from (4.22) read:

$$\chi_j = \frac{1}{2}\eta^l{}_j \hat{\mathscr{D}}_l \hat{\mathscr{K}} + \frac{1}{2}\hat{\mathscr{D}}_j \hat{\mathscr{A}} - 2 * \varpi \hat{\mathscr{R}}_j, \tag{G.29}$$

$$\psi_j = 3\eta^l{}_j \hat{\mathscr{D}}_l * \varpi^2, \tag{G.30}$$

$$X_{ij} = \frac{1}{2}\eta^l{}_j \hat{\mathscr{D}}_l \hat{\mathscr{R}}_i + \frac{1}{2}\eta^l{}_i \hat{\mathscr{D}}_j \hat{\mathscr{R}}_l, \tag{G.31}$$

$$\Psi_{ij} = \hat{\mathscr{D}}_i \hat{\mathscr{D}}_j * \varpi - \frac{1}{2} a_{ij} \hat{\mathscr{D}}_l \hat{\mathscr{D}}^l * \varpi - \eta_{ij} \frac{1}{\Omega} \hat{\mathscr{D}}_t * \varpi^2. \tag{G.32}$$

Integrability conditions dictate that the components of energy-momentum tensor are dual to these and are given by (4.29) and (4.30). For sake of completeness, we write some of the Carrollian quantities in the complex coordinates $(\zeta, \bar{\zeta})$ with boundary metric at $\mathscr{I}$ in conformally flat form $\mathrm{d}l^2$ given in (F.7):

$$\varphi_\zeta = \partial_t \frac{b_\zeta}{\Omega} + \hat{\partial}_\zeta \ln \Omega, \quad \varphi_{\bar{\zeta}} = \partial_t \frac{b_{\bar{\zeta}}}{\Omega} + \hat{\partial}_{\bar{\zeta}} \ln \Omega, \tag{G.33}$$

$$\theta = -\frac{2}{\Omega}\partial_t \ln P, \quad *\varpi = \frac{\mathrm{i}\Omega P^2}{2}\left(\hat{\partial}_\zeta \frac{b_{\bar{\zeta}}}{\Omega} - \hat{\partial}_{\bar{\zeta}} \frac{b_\zeta}{\Omega}\right) \tag{G.34}$$

with

$$\hat{\partial}_\zeta = \partial_\zeta + \frac{b_\zeta}{\Omega}\partial_t, \quad \hat{\partial}_{\bar{\zeta}} = \partial_{\bar{\zeta}} + \frac{b_{\bar{\zeta}}}{\Omega}\partial_t. \tag{G.35}$$

Also,

$$\hat{\mathscr{K}} = \hat{K} + \hat{\nabla}_k \varphi^k, \quad \hat{\mathscr{A}} = \hat{A} - *\varphi, \quad \text{where} \tag{G.36}$$

$$\hat{K} = P^2\left(\hat{\partial}_{\bar{\zeta}}\hat{\partial}_\zeta + \hat{\partial}_\zeta\hat{\partial}_{\bar{\zeta}}\right)\ln P, \quad \hat{A} = \mathrm{i}P^2\left(\hat{\partial}_{\bar{\zeta}}\hat{\partial}_\zeta - \hat{\partial}_\zeta\hat{\partial}_{\bar{\zeta}}\right)\ln P, \tag{G.37}$$

$$*\varphi = \mathrm{i}P^2\left(\hat{\partial}_\zeta \varphi_{\bar{\zeta}} - \hat{\partial}_{\bar{\zeta}}\varphi_\zeta\right), \tag{G.38}$$

$$\hat{\nabla}_k \varphi^k = P^2\left[\hat{\partial}_\zeta \partial_t \tfrac{b_{\bar{\zeta}}}{\Omega} + \hat{\partial}_{\bar{\zeta}} \partial_t \tfrac{b_\zeta}{\Omega} + \left(\hat{\partial}_\zeta\hat{\partial}_{\bar{\zeta}} + \hat{\partial}_{\bar{\zeta}}\hat{\partial}_\zeta\right)\ln\Omega\right]. \tag{G.39}$$

Appendix H

Carrollian fluids, Schwarzschild-Taub-NUT

In appendix G, we showed that all information about the boundary Carrollian fluid is in the geometric data of $\mathscr{S}$, i.e., $a_{ij}(t,\mathbf{x})$, $b_i(t,\mathbf{x})$ and $\Omega(t,\mathbf{x})$. Let us try to reconstruct the Kerr-TaubNUT solution starting with some given Carrollian data.

To do this, we will first find the Carrollian data for the simple case of Schwarzschild geometry, and then show that similar data can give us the complete Kerr-Taub-NUT solution. Let us expand the resummed metric (4.27) and write each component of the metric separately:

$$\begin{aligned}
\mathrm{d}s^2_{res.flat} &= -2(\Omega \mathrm{d}t - \boldsymbol{b})\left(\mathrm{d}t + r\boldsymbol{\alpha} + \frac{r\theta\Omega}{2}\mathrm{d}t\right) + r^2 a_{ij}\mathrm{d}x^i \mathrm{d}x^j + \boldsymbol{s} + \frac{(\Omega \mathrm{d}t - \boldsymbol{b})^2}{\rho^2}(8\pi G\epsilon r + c \star \omega) \\
&= -2(\Omega \mathrm{d}t - b_\zeta \mathrm{d}\zeta - b_{\bar\zeta}\mathrm{d}\bar\zeta)\left(\mathrm{d}t + r(\alpha_\zeta \mathrm{d}\zeta - \alpha_{\bar\zeta}\mathrm{d}\bar\zeta) + \frac{r\theta\Omega}{2}\mathrm{d}t\right) + \frac{2r^2}{P^2}\mathrm{d}\zeta \mathrm{d}\bar\zeta \\
&\qquad + \boldsymbol{s} + \frac{(\Omega \mathrm{d}t - b_\zeta \mathrm{d}\zeta - b_{\bar\zeta}\mathrm{d}\bar\zeta)^2}{\rho^2}(8\pi G\epsilon r + c\star\omega) \\
&= A(\Omega \mathrm{d}t - L\mathrm{d}Z - iD\mathrm{d}\Phi)^2 + \frac{2\rho^2}{P^2}(\mathrm{d}Z^2 + Z^2 \mathrm{d}\Phi^2) - 2(\Omega \mathrm{d}t - L\mathrm{d}Z \\
&\qquad - iD\mathrm{d}\Phi)\left(\mathrm{d}r + B\mathrm{d}Z + iQ\mathrm{d}\Phi + \frac{r\theta\Omega}{2}\mathrm{d}t\right)
\end{aligned} \tag{H.1}$$

where we have substituted the complex coordinates $(\zeta, \bar\zeta)$ by

$$\zeta = Z\exp\{i\Phi\} \qquad \bar\zeta = Z\exp\{-i\Phi\} \tag{H.2}$$

Also, the quantities appearing in the last line of (H.1) are:

$$\begin{aligned}
A(r,z,\phi) &= \frac{8\pi G\epsilon r + c * \varpi}{\rho^2} - \hat{\mathscr{K}}, \\
L(z,\phi) &= b_\zeta e^{i\Phi} + b_{\bar\zeta}e^{-i\Phi}, \\
D(z,\phi) &= Z\left(b_\zeta e^{i\Phi} - b_{\bar\zeta}e^{-i\Phi}\right), \\
B(r,z,\phi) &= (r\alpha_\zeta + i\hat{\mathscr{D}}_\zeta * \varpi)e^{i\Phi} + (r\alpha_{\bar\zeta} + i\hat{\mathscr{D}}_{\bar\zeta} * \varpi)e^{-i\Phi}, \quad \text{and} \\
Q(r,z,\phi) &= Z\left((r\alpha_\zeta + i\hat{\mathscr{D}}_\zeta * \varpi)e^{i\Phi} - (r\alpha_{\bar\zeta} + i\hat{\mathscr{D}}_{\bar\zeta} * \varpi)e^{-i\Phi}\right).
\end{aligned} \tag{H.3}$$

Here c, $*\varpi$, ρ^2, $\hat{\mathscr{K}}$, α_ζ, $\hat{\mathscr{D}}$ are given in (G.28) ,(G.33), (4.28), (G.36) and (4.25) respectively. Also since $*\varpi$ is a scalar of weight-1, and hence it's derivative $\hat{\mathscr{D}}_i * \varpi$ is taken using (G.21). With this, we can write the components of the resummed metric as

$$\begin{aligned}
\mathrm{d}s^2_{res.flat} &= \Omega^2(A - r\theta)\mathrm{d}t^2 - 2\Omega \mathrm{d}r\mathrm{d}t + \mathrm{d}Z^2\left(\frac{2\rho^2}{P^2} + L(AL + 2B)\right) + 2L\mathrm{d}r\mathrm{d}Z \\
&+ \mathrm{d}\Phi^2\left(\frac{2\rho^2 Z^2}{P^2} - D(AD - 2Q)\right) + \Omega \mathrm{d}t\mathrm{d}Z\left(L(r\theta - 2A) - 2B\right) + 2iD\mathrm{d}r\mathrm{d}\Phi \\
&\qquad + i\Omega \mathrm{d}t\mathrm{d}\Phi\left(D(r\theta - 2A) - 2Q\right) + 2i\mathrm{d}z\mathrm{d}\Phi(ALD + LQ + DB)
\end{aligned} \tag{H.4}$$

Let us compare this metric with the standard Schwarzschild metric in Eddington-Finkelstein coordinates, (u, r, Z, Φ) to get the unknown pieces Ω, b_i, P and ϵ:

$$\mathrm{d}s^2 = -f(r)\mathrm{d}u^2 - 2\mathrm{d}r\mathrm{d}u + \frac{2r^2}{\left(1+\frac{Z^2}{2}\right)^2}(\mathrm{d}Z^2 + Z^2\mathrm{d}\Phi^2) \tag{H.5}$$

where u is the retarded time coordinate. Clearly, the g_{tr}, g_{rZ}, and $g_{r\Phi}$ components require $\Omega = 1$, $L = D = 0$ respectively. This implies that $b_\zeta = b_{\bar{\zeta}} = 0$. Substituting this in g_{tZ} and $g_{t\Phi}$ components implies that $B = Q = 0$. Also note that from (G.33) and (G.34) we have $\varphi_i = 0$ and $*\varpi = 0$. From (4.25) this implies $\alpha_i = 0$. Now comparing the g_{ZZ} and $g_{\Phi\Phi}$ components tell us that $P = 1 + \frac{Z^2}{2}$. Finally, substituting this in (G.36) tells us that $\hat{\mathscr{K}} = 1$ and hence, $A = -(1 - \frac{8\pi G\epsilon}{r})$. Comparing the g_{tt} component now demands that $4\pi G\epsilon = M$, where M is the mass of the Schwarzschild black hole. Hence, the boundary data for Schwarzschild geometry is:

$$\begin{aligned}
\Omega &= 1 \\
b_\zeta &= b_{\bar{\zeta}} = 0 \\
P &= 1 + \frac{\zeta\bar{\zeta}}{2} = 1 + \frac{Z^2}{2} \\
\epsilon &= \frac{M}{4\pi G}
\end{aligned} \tag{H.6}$$

Following a similar procedure for the Taub-NUT metric (A.1) in (Z, Φ)-coordinates:

$$\mathrm{d}s^2 = -f(r)\left(\mathrm{d}t + \frac{2\nu}{1+\frac{Z^2}{2}}Z^2\mathrm{d}\Phi\right)^2 - \frac{4\nu Z^2}{1+\frac{Z^2}{2}}\mathrm{d}r\mathrm{d}\Phi - 2\mathrm{d}r\mathrm{d}t + \frac{2(r^2+\nu^2)}{\left(1+\frac{Z^2}{2}\right)^2}(\mathrm{d}Z^2 + Z^2\mathrm{d}\Phi^2) \tag{H.7}$$

where

$$f(r) = \frac{r^2 - 2mr - \nu^2}{r^2 + \nu^2} \tag{H.8}$$

Once again, the g_{tr} and g_{rZ} components require $\Omega = 1$ and $L = 0$ respectively. The $g_{r\Phi}$ component requires

$$D = -\frac{2\nu i Z^2}{1+\frac{Z^2}{2}} = -\frac{2\nu i\zeta\bar{\zeta}}{1+\frac{\zeta\bar{\zeta}}{2}} \tag{H.9}$$

Therefore,

$$b_\zeta = -\frac{\nu i Z}{2(1+\frac{Z^2}{2})}e^{-i\Phi} = -\frac{\nu i\bar{\zeta}}{2(1+\frac{\zeta\bar{\zeta}}{2})}, \qquad b_{\bar{\zeta}} = \frac{\nu i Z}{2(1+\frac{Z^2}{2})}e^{i\Phi} = \frac{\nu i\zeta}{2(1+\frac{\zeta\bar{\zeta}}{2})} \tag{H.10}$$

substituting this in (G.34), gives $\varpi = \nu$. This implies that $B = Q = 0$. Then, the g_{tZ} and $g_{t\Phi}$ components are satisfied trivially provided $P = 1 + \frac{Z^2}{2}$. This also varifies with g_{ZZ} and $g_{Z\Phi}$ components. Finally, substituting this in (G.36) tells us that $\hat{\mathscr{K}} = 1$ and hence, $A = -(1 - \frac{8\pi G\epsilon}{r})$. Comparing the g_{tt} we once again get $4\pi G\epsilon = m$. Hence, the boundary data corresponding to Taub-NUT is:

$$\begin{aligned}
&\Omega = 1 \\
&b_\zeta = -\frac{\nu i\bar{\zeta}}{2(1+\frac{\zeta\bar{\zeta}}{2})}, \qquad b_{\bar{\zeta}} = \frac{\nu i\zeta}{2(1+\frac{\zeta\bar{\zeta}}{2})} \\
&P = 1 + \frac{\zeta\bar{\zeta}}{2} = 1 + \frac{Z^2}{2} \\
&\epsilon = \frac{M}{4\pi G}
\end{aligned} \tag{H.11}$$

Following similar procedure for Kerr metric, we can show that we can reach the whole family of Kerr-Taub-NUT stationary Ricci flat black holes by setting the geometric data to be: $P(\boldsymbol{x})$, $b_i(\boldsymbol{x})$ and $\Omega = 1$. Starting with this data, we can recover the family of solutions:

$$\begin{aligned}
\mathrm{d}s^2_{\text{perf. fl.}} \;=\; & -\frac{\Delta_r}{\rho^2}\left(\mathrm{d}t + \frac{2}{P}\left(n - \frac{a}{P} + \frac{\nu}{2P}\left(1-|K|\right)Z^2\right)Z^2\mathrm{d}\Phi\right)^2 + \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 \\
& + \frac{2\rho^2}{P^2}\mathrm{d}Z^2 + \frac{2Z^2}{\rho^2P^2}\left(\left(Ka + \nu\left(1-|K|\right)\right)\mathrm{d}t - \left(r^2 + (n-a)^2\right)\mathrm{d}\Phi\right)^2
\end{aligned} \tag{H.12}$$

with

$$\begin{aligned}
P = 1 + \frac{K}{2}Z^2, \quad \rho^2 = r^2 + \left(n + a - \frac{2a}{P} + \frac{\nu}{P}\left(1-|K|\right)Z^2\right)^2, \\
\Delta_r = -2Mr + K\left(r^2 + a^2 - n^2\right) + 2\nu(n-a)(|K|-1).
\end{aligned} \tag{H.13}$$

Here we have moved from Eddington–Finkelstein to Boyer–Lindquist coordinates by,

$$\mathrm{d}t \to \mathrm{d}t - \frac{r^2 + (n-a)^2}{\Delta_r}\mathrm{d}r\,, \quad \mathrm{d}\Phi \to \mathrm{d}\Phi - \frac{Ka + \nu(1-|K|)}{\Delta_r}\mathrm{d}r \tag{H.14}$$

Here, the $n,\ a,\ l$ parameters are the NUT, rotation and flat-rotation parameters respectively that one sees in Kerr-TaubNUT family (see [1]). Also, $K = 0, \pm 1$ depending on whether the $\mathcal{S}$ corresponds to sphere S^2, the Euclidean plane E_2 or hyperbolic plane H_2. We also show that the 'inverse velocity' parameter is

$$\boldsymbol{b} = \frac{\mathrm{i}}{P}\left(n - \frac{a}{P} + \frac{\nu}{2P}\left(1-|K|\right)\zeta\bar{\zeta}\right)\left(\bar{\zeta}\mathrm{d}\zeta - \zeta\mathrm{d}\bar{\zeta}\right). \tag{H.15}$$

and the energy density is always identified with bulk mass $M = 4\pi G\epsilon$.

Appendix I

Geroch dual

For a timelike Killing vector, ∂_t with the Ricci-flat resummed metric (4.27), the one form is given by g_{ti},

$$\underline{\xi} = \Omega\left(\Omega(A - r\theta)\mathrm{d}t - \mathrm{d}r + ((r\theta - 2A)L - 2B)\frac{\mathrm{d}Z}{2} + i((r\theta - 2A)D - 2Q)\frac{\mathrm{d}\Phi}{2}\right) \tag{I.1}$$

Then we can write

$$\mathrm{d}\underline{\xi} = \mathrm{d}r \wedge \partial_r \xi + \mathrm{d}Z \wedge \partial_Z \xi + \mathrm{d}\Phi \wedge \partial_\Phi \xi \tag{I.2}$$

We can use this to find the vorticity of ξ (4.34). For the case of $\Omega = 1$, we have verified that we recover the results for Kerr-TaubNUT metrices (at least for the Schwarzschild and Taub-NUT cases), however these are yet to be verified and we will not display the results for Killing vorticity one-form and scalar.

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