Turbulence and Navier-Stokes: Existence, Smoothness and Beyond. Insights from Soft Matter Physics

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1 Introduction

Physicists often only deal with 'ideal' systems that are as simple and 'pure' as possible, but soft matter has shown that even 'untidy' physical systems like biological materials and colloidal solutions can be successfully described in general terms.[1] The softness of soft-matter implies that it is easily driven far from equilibrium, making them the perfect place to study velocity fluctuations and turbulence,[2] which is probably the last unsolved problem in Newtonian physics.[3]

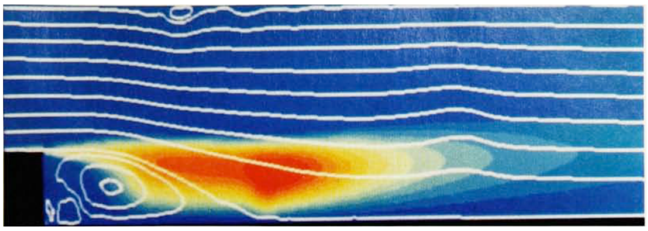

Figure 1. Stationary turbulent flow over a step (in black)[4]

In the continuum limit (small mean free path), Newton's second law for a fluid with an unknown velocity vector $\boldsymbol{v}$ and pressure $p(\boldsymbol{x}, t)$, can be written as

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}.\nabla)\boldsymbol{v} = -\frac{1}{\rho}\nabla p + \nu\Delta\boldsymbol{v} + \boldsymbol{f}(\boldsymbol{x}, t) \qquad (1)$$

where $\nu > 0$ is kinematic viscosity and $\boldsymbol{f}(\boldsymbol{x}, t)$ represents external volumetric forces. For an incompressible fluid

$$\nabla.\boldsymbol{v} = 0 \qquad (2)$$

(1) and (2) combined are called the Navier-Stokes equations.

Even though these basic equations of macroscopic classical physics have been well-known for centuries, the simplest solutions (corresponding to $\boldsymbol{f}(\boldsymbol{x}, t) = 0$) for these are still unknown due to the non-linearity of the equations (some specific solutions are known, however, it is yet to be proven that given any arbitrary initial velocity, solutions exist for arbitrarily large time scale). How can we deeply understand the general solution (which explains turbulent flows), if we don't even know that the fluid equations have solutions?[5] This made Clay Mathematical Institute declare Navier-Stokes existence (or non-existence) and smoothness as one of the seven Millennium prize problems.[6]

This essay, therefore, deals with two problems: solutions to Navier-Stokes equations (which is more mathematical than physical) and turbulence. We will try to motivate that soft matter physics can provide valuable insights into what these solutions should look like (and how one should go about looking for them) and their interpretation, so that we can construct a theoretical model for the statistics of turbulent flows.

2 Current understanding of turbulence and solutions to Navier-Stokes equation

Turbulence is a ubiquitous phenomenon in nature, from mixing of cream in a coffee cup to the formation of galaxies. It is characterized by the chaotic changes in the pressure and flow velocity in fluids which are present in almost all flows, natural or man-made.[7] Advances in key issues such as energy generation, pollution mitigation, and climate change, as well as progress in several fields of fundamental science from astrophysics to geophysics, are limited by the lack of understanding of the physics of turbulence.[8] However, to this date, there is no consensus on what should constitute as a solution to the turbulence problem. From an engineering perspective, we are interested in mean properties of these "random" flows, while from the physics point of view, it is crucial to understand the nonlinear physical processes responsible for those mean properties as well as the details of motions across the broad range of excited scales.[9]

There are four basic reasons that make turbulence a hard concept to understand: Randomness, Eddy viscosity, cascade and scaling.[4]

Randomness: According to the classical interpretation, turbulent flows appear random because intrinsic instabilities of the flow amplify the thermal/mechanical fluctuations. Despite a lot of improvement in our understanding of such chaotic behaviour in dynamical systems, we still don't understand high Reynolds number flows which have large attractors.

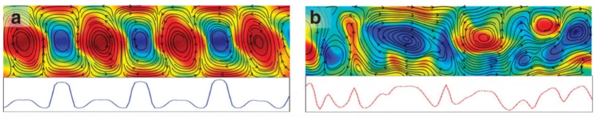

Figure 2. (a) A highly ordered flow and (b) active turbulence. Lower panels show the enstrophy signal along the channel[10]

Eddy viscosity: Taking analogy from molecular motion, the dynamics on hydrodynamic scale become diffusive when the motion are on widely separated scales (this is, for example, the reason why a drop of dye diffuses at rates much higher than its molecular rates), however, in turbulent flows, dominant interactions are usually contiguous making viscosity models not perfectly correct models to describe them.

Cascading: Large eddies which are unstable are formed due to the forces driving the flow. These eddies are unstable and form smaller eddies, which themselves become unstable and the process continues until molecular viscosity can suppress further cascading. Such cascading effects make it hard to understand the dynamics of a particular eddy.

Scaling: At both very large and small length scales (compared to Kolmogorov dissipation scale, $l_d = (v_{mol}/\mathcal{E})^{1/4}$ where $\mathcal{E}$ is the energy flux), scale invariance breaks down, making it hard to understand turbulence at these length scales.

3 Implications from computational soft matter

Our usual statistical mechanics approaches only give equilibrium results, while the purpose of hydrodynamic equations is to deal with situations where equilibrium is only attained locally. Hence the nonlinear coupling mechanisms for response and dynamical behavior of flowing soft matter, force us to resort to coarse-grained descriptions that express a large number of degrees of freedom through a much smaller number of effective degrees of freedom whilst retaining the correct overall physical behaviour.[11] These computational methods then provide a way to model the stochastic nature of turbulence and compare it with actual observations. This power of computational soft condensed matter was recognised as early as 1946, when John von Neumann remarked, "computational fluid dynamics would make experimental fluid dynamics obsolete"[12]

Specialised simulation methods (mesoscopic modelling) which include only the essential details of interactions by say collective collision satisfying local conservation or hydrodynamics on a lattice, have to reproduce Navier-Stokes hydrodynamics asymptotically. Hence, approaches like Brownian dynamics, $O(N^3)$ and Chebyshev polynomial approximation, $0(\mathrm{N}^{2.5})$ have helped us to understand the constitutive relations between the transport properties and implementation of boundary conditions on the real system on a macroscopic level.[13] Some commonly used mesoscopic modelling methods are: dissipative particle dynamics (DPD), multi-particle collision dynamics (MPC), lattice Boltzmann method (LBZ) and high-performance computing.

In the following examples I outline the potential of these techniques to understand turbulence and solutions to Navier-Stokes equations: **DPD** has been used to model different dynamic regimes of polymer chains undergoing hydrodynamic interactions at small length scale and repetitional dynamics at long lengths where chains are entangled and feel topological constraints.[14] **MPC**: Non-equilibrium flows of microfluidic droplets in a Hele-Shaw geometry can be modelled within 2D-MPC confirming validity of far-field approximation of hydrodynamic interactions at high density and longitudinal motion induced by boundary conditions on flow field confirming the ability of MPC to investigate non-equilibrium physics and develop novel methods in statistical physics.[15] **LBM** has a plethora of boundary conditions, like the bounce-back boundary condition which has no conventional counterpart in Navier-Stokes hinting towards improving/correcting the Navier-Stokes equation for our understanding of turbulence.[16–19] Attempts have also been made to understand Brownian motion and thermal fluctuations using LBM[20]

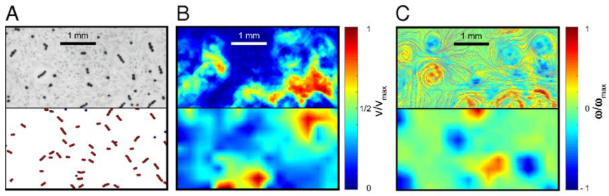

Figure 3. Turbulence: Experimental(top) and MPC(bottom)[21]

4 Beyond Navier-Stokes equations

Navier-Stokes equations are only valid in the continuum limit, that is, when the gradients of velocity, temperature and density are sufficiently small so that we are not very far away from the equilibrium. There are many simple experiments[22] (known as far back as the 1880s) which couldn't be explained by Navier-Stokes equations. Hence one can wonder if models that are valid despite such highly non-equilibrium fluxes might be able to shed any light on the nature of turbulence.

Soft matter systems like complex fluids and plasma have known to exhibit turbulence whose dynamics are described by more complex equations than Navier-Stokes flows. These systems might be a gateway to the 'beyond Navier-Stokes' hydrodynamics and understanding turbulence. One such system of quickly growing scientific interests is turbulence in Active fluids, an active system consisting of self-propelled particles with mesoscale turbulent motion. Such self-sustained "active turbulence" can have profound effects on nutrient mixing and molecular transport in microbiological systems.[23] Due to the internal microscopic forcing, active systems have turbulence at small Reynolds number as opposed to very large Reynolds number of ordinary turbulent fluids.[24] It has been shown that due to internal instabilities, in addition to the convective non-linearities of Navier-Stokes type, higher-order non-linearities are also present in active fluids, providing additional freedom for the system to 'self-tune' into a critical state.[25] This if true, will imply that turbulence can indeed have non-universal behaviour. Numerous computational models based on active matter have been proposed, some of which are successful in describing the underlying out-of equilibrium character, multiscale nature, nonlinearity, and multibody interactions.[26]

5 Conclusions

In this article, I have pointed out the importance and difficulties in finding solutions to Navier-Stokes equations and how an understanding of turbulence can require beyond Navier-Stokes hydrodynamics. We saw that computational soft matter physics and modelling has provided many new insights into both turbulence and beyond Navier-Stokes physics due to the wide range of tunable parameters, making predictions which are very close to experimental results. With an increased interest in topics like active matter, we can hopefully understand the ever eluding problem of turbulence in the next couple of decades. It is, however, unclear whether soft matter physics can be used to solve the Millennium problem, because they work in either small-time ($[0, T)$ instead of $[0, \infty)$) or small-velocity limit.

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